Option A Relativity

A1 The beginnings of relativity

It is said that Albert Einstein, as a boy, asked himself what would happen if he held a mirror in front of himself and ran forward at the speed of light. With respect to the ground, the mirror would be moving at the speed of light. Rays of light leaving young Einstein's face would also be moving at the speed of light relative to the ground. This meant that the rays would not be moving relative to the mirror, and hence there should be no reflection in it. This seemed odd to Einstein. He expected that looking into the mirror would not reveal anything unusual. Some years later, Einstein would resolve this puzzle with a revolutionary new theory of space and time, the theory of special relativity.

Learning objectives

- Use reference frames.
- Understand Galilean relativity with Newton's postulates for space and time.
- Understand the consequences of Maxwell's theory for the speed of light.
- Understand how magnetic effects are a consequence of relativity.

A1.1 Reference frames

In a physics experiment, an observer records the time and position at which **events** take place. To do that, she uses a **reference frame**.

> A reference frame is a set of coordinate axes and a set of clocks at every point in space. If this set is not accelerating, the frame is called an **inertial reference frame** (Figure A.1).

So if the 'event' is a lightning strike, an observer will look at the reading of the clock at the point where lightning struck and record that reading as the time of the event. The coordinates of the strike point give the position of the event in space. So, in Figure **A.2**, lightning strikes at time $t = 3$ s and position $x = 60$ m and $y = 0$. (We are ignoring the z coordinate.)

The same event can also be viewed by another observer in a different frame of reference. Consider, therefore, the following situation involving one observer on the ground and another who is a passenger on a train.

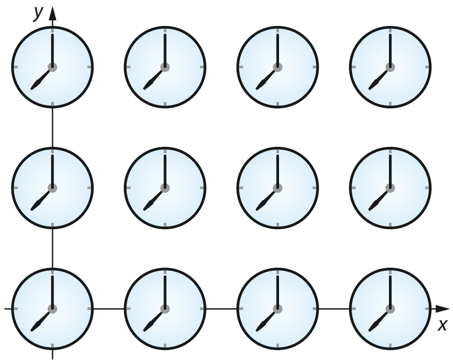


Figure A.1 A two-dimensional reference frame. There are clocks at every point in space (only a few are shown here). All clocks show the same time.

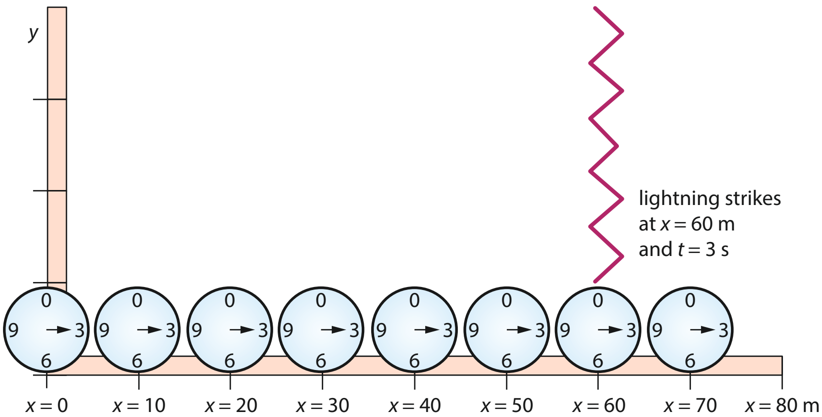


Figure A.2 In this frame of reference the observer decides that lightning struck at time $t = 3$ s and position $x = 60$ m.

In Figure **A.3**, a train moves past the observer on the ground (represented by the thin vertical line) at time $t = 0$, and is struck by lightning 3 s later. The observer on the train is represented by the thick vertical line. The train is travelling at a velocity of $v = 15\,\mathrm{m\,s^{-1}}$ as far as the ground observer is concerned.

Let us see how the two observers view various events along the trip. Assume first that when the clocks carried by the two observers show zero, the origins of the rulers carried by the observers coincide, as shown in Figure **A.3**. The stationary observer uses the symbol x to denote the distance of an event from the origin of his ruler. The observer on the train uses the symbol x'.

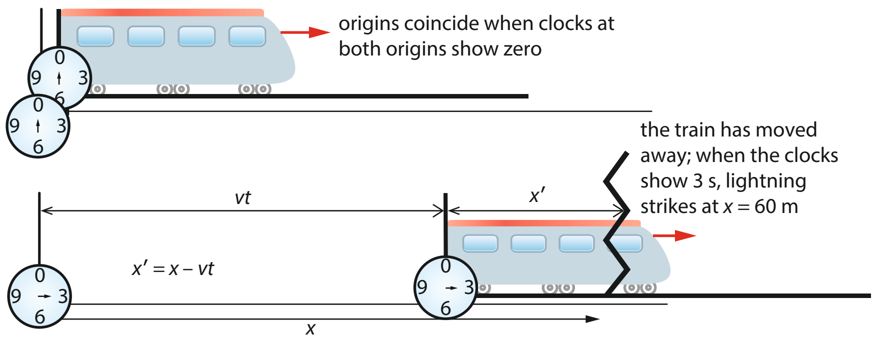


Figure A.3 The origins of the two frames of reference coincide when clocks in both frames show zero. The origins then separate.

Lightning strikes a point on the train. The observer on the train measures the point where the lightning strikes and finds the answer to be x'. The ground observer measures that the lightning struck at a distance x from his origin.

$$x' = x - vt$$

$$t' = t$$

In time t s the train moves forward a distance vt. These equations express the relationship between the coordinates of the same event as viewed by two observers who are in relative motion.

Thus, to the event in Figure **A.3** ('lightning strikes') the ground observer assigns the coordinates $x = 60$ m and $t = 3$ s. The observer on the train assigns to this same event the coordinates $t' = 3$ s and $x' = 60 - 15 \times 3 = 15$ m.

We are assuming here what we know from everyday experience (a guide that, as we will see, may not always be reliable): that two observers always agree on what the time coordinates are; in other words, time is common to both observers. Or, as Newton wrote,

> Absolute, true and mathematical time, of itself, and from its own nature, flows equably without any relation to anything external.

Of course, the observer on the train may consider herself at rest and the ground below her to be moving away with velocity $-v$. It is impossible for one of the observers to claim that he or she is 'really' at rest and that the other is 'really' moving. There is no experiment that can be performed by, say, the observer on the train that will convince her that she 'really' moves (apart from looking out of the window). If we consider instead a space station and a spacecraft in outer space as our two frames, even looking out of the window will not help. Whatever results the observer on the train gets out of her experiments, the ground observer also gets out of the same experiments performed in his ground frame of reference. The equations

$$x' = x - vt$$

$$t' = t$$

are called **Galilean transformation** equations, in honour of Galileo (Figure **A.4**): the relationship between the coordinates of an event when one frame moves past the other with uniform velocity in a straight line. Both observers are equally justified in considering themselves to be at rest, and the descriptions they give are equally valid.

Galilean relativity has an immediate consequence for the **law of addition of velocities**. Consider a ball that rolls with velocity u' as measured by the observer on the train. Again assume that the two frames, train and ground, coincide when $t = t' = 0$ and that the ball first starts rolling when $t' = 0$. Then, after time t' the position of the ball is measured to be at $x' = u't'$ by the observer on the train (Figure **A.5**).

The ground observer records the position of the ball to be at $x = x' + vt = (u' + v)t$ (recall $t = t'$), so as far as the ground observer is concerned the ball has a velocity (distance/time) given by

$$u = u' + v$$

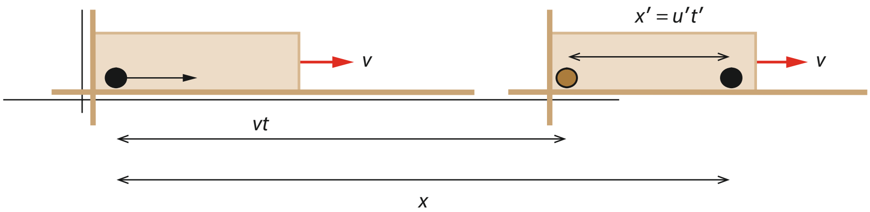


Figure A.5 An object rolling on the floor of the 'moving' frame appears to move faster as far as the ground observer is concerned.

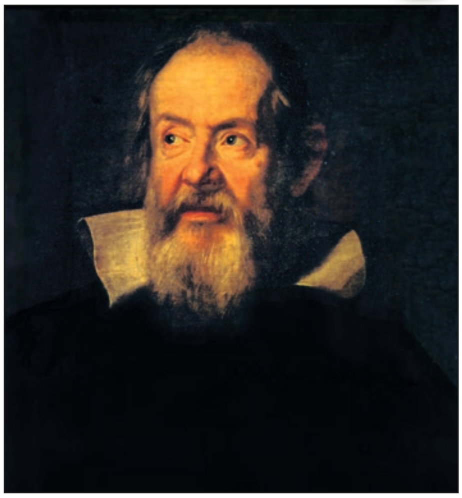


Figure A.4 Galileo Galilei (1564–1642).

Worked example

A.1 A ball rolls on the floor of a train at $2\,\mathrm{m\,s^{-1}}$ (with respect to the floor). The train moves with respect to the ground **a** to the right at $12\,\mathrm{m\,s^{-1}}$, **b** to the left at $12\,\mathrm{m\,s^{-1}}$. What is the velocity of the ball relative to the ground?

a The velocity is $14\,\mathrm{m\,s^{-1}}$.

b The velocity is $-10\,\mathrm{m\,s^{-1}}$.

This apparently foolproof argument presents problems, however, if we replace the rolling ball in the train by a beam of light moving with velocity $c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$ as measured by the observer on the train. Using the formula above implies that light would be travelling at a higher speed relative to the ground observer. At the end of the 19th century, considerable efforts were made to detect variations in the speed of light depending on the state of motion of the source of light. The experimental result was that no such variations were detected!

A1.2 Maxwell and the speed of light

In 1864, James Clerk Maxwell corrected an apparent flaw in the laws of electromagnetism by introducing his famous 'displacement current' term into the electromagnetic equations. The result is that a changing electric flux produces a magnetic field just as a changing magnetic flux produces an electric field (as Faraday had discovered earlier). An immediate conclusion was that accelerated electric charges produce a pair of self-sustaining electric and magnetic fields at right angles to each other, which eventually decouple from the charge and move away from it at the speed of light. Maxwell discovered **electromagnetic waves** and thus demonstrated the electromagnetic nature of light.

One prediction of the Maxwell theory was that the speed of light is a **universal constant**. Indeed, Maxwell was able to show that the speed of light is given by the expression

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

where the two constants are the electric permittivity and magnetic permeability of free space (vacuum): two constants at the heart of electricity and magnetism.

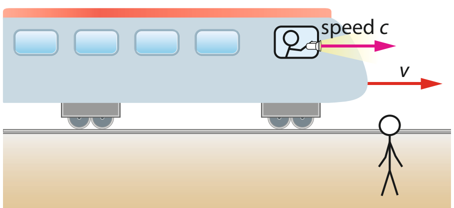


Figure A.6 An observer on the train measures the speed of light to be c. An observer on the ground would then measure a different speed, $c+v$, according to Galileo.

> Maxwell's theory predicts that the speed of light in vacuum does not depend on the speed of its source.

This is in direct conflict with Galilean relativity. According to Galilean relativity, if the speed of light takes on the value c in the 'train' frame of reference then it will have the value $c+v$ in the 'ground' frame of reference (Figure **A.6**).

The situation faced by Einstein in 1905 was that the Galilean transformation equations (which were perfectly compatible and consistent with Newtonian mechanics) did not seem to be compatible with Maxwell's theory. It turned out that, a bit before 1905, the Dutch physicist Hendrik Lorentz (Figure **A.7**) was also thinking about the same problem.

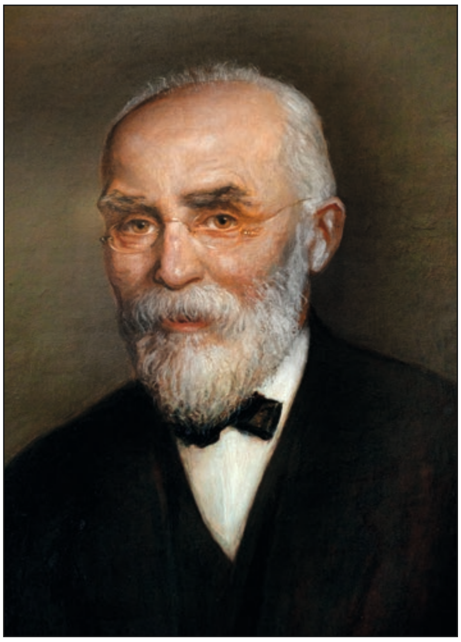


Figure A.7 Hendrik Lorentz (1853–1928).

He too realised that the Maxwell theory was not compatible with the Galilean transformation equations. He set about trying to find the simplest set of transformation equations that would be compatible with Maxwell's theory. Lorentz's answer was a strange-looking set of equations, the **Lorentz transformation** equations (Table **A.1**):

Galileo	Lorentz
$x' = x - vt$	$x' = \dfrac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$
$t' = t$	$t' = \dfrac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}$

Table A.1 Galilean and Lorentz transformations.

As we will soon see, these equations predict two brand new phenomena, which we will discuss in detail later: **length contraction** (a moving object is measured to have a smaller length than when it is at rest) and **time dilation** (moving clocks run slow). But the new problem now was that these new equations were not compatible with Newtonian mechanics! So there were two choices available (Table **A.2**).

Choice A	Choice B
Accept Newtonian mechanics and the Galilean equations and then modify Maxwell's theory	Accept Maxwell's theory and the Lorentz equations and then modify Newtonian mechanics

Table A.2 The choices Einstein was confronted with.

Einstein's choice was B. He realised early on that all kinds of puzzles arise when one looks at electric and magnetic phenomena from two different inertial reference frames. These could only be resolved if one made choice B.

A1.3 Electromagnetic puzzles

Imagine a wire at rest in a lab, in which a current flows to the left. This means that the electrons in the wire are moving with a drift speed v to the right (Figure **A.8**). The wire also contains positive charges that are at rest. The average distance between the electrons and that between the positive charges are the same. The net charge of the wire is zero, so an observer at rest in the lab measures zero electric field around the wire. Now imagine a positive charge q that moves with the same velocity as the electrons in the rod. The observer in the lab has no doubt that there will be a **magnetic** force on this charge. The magnetic field at the position of the charge is out of the page and so there will be a magnetic force repelling the charge q from the wire (use the right-hand rule for magnetic force).

Now consider things from the point of view of another observer who moves along with the charge q. For this observer, the charge q and the electrons in the rod are at rest, but the positive charges in the rod move to the left with speed v. These positive charges do produce a magnetic field, but the magnetic force on q is zero since $F = qvB$ and the speed of the charge is zero.

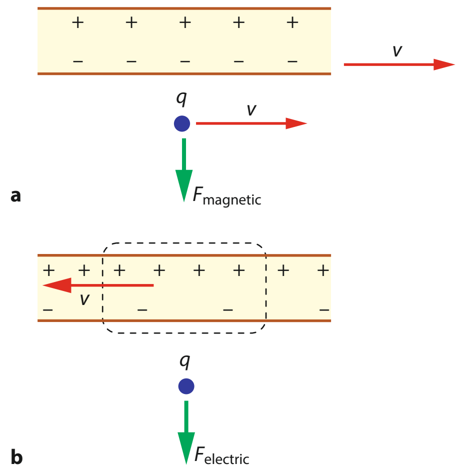


Figure A.8 **a** An observer at rest with respect to the wire measures a magnetic force on the moving charge. **b** An observer moving along with the charge will measure an electric force.

Relativity and reality

Relativity says that the details of an experiment may appear different to different observers in relative motion. This does not mean that different observers experience a different 'reality'. The physically significant aspects of the experiment are the same for both observers.

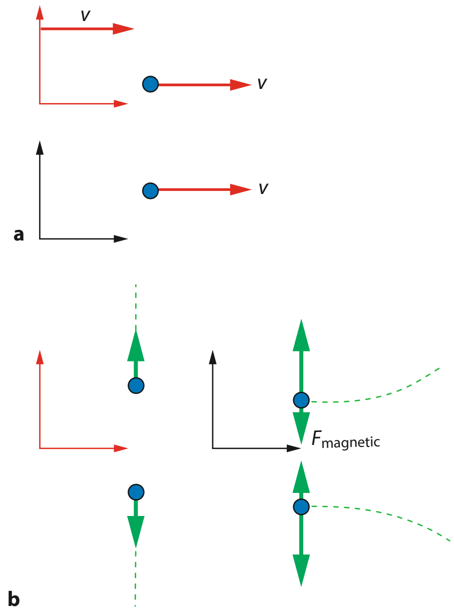


Figure A.9 **a** Two protons move past an observer with the same velocity *v*. **b** The electric and magnetic forces on the charges look different in each reference frame.

Now, if the charge q feels a force according to one observer, other observers must reach similar conclusions. So the puzzle is: where does the force on the charge q come from? Notice that in Figure **A.8b** we have made the distance between positive charges smaller and the distance between electrons larger. This has to be the case if Lorentz is right about length contraction. The separation between the positive charges is a distance that moves past the observer and so has to get smaller. The separation between the electrons used to be a 'moving' distance so, now that it has stopped, it has to be bigger. The effect of this is that now, as far as the charge q is concerned, there is more positive charge than negative charge in the wire near it. There will then be an **electric** force of repulsion. We have learned that, as a result of length contraction, the force that an observer calls magnetic in one reference frame may be an electric force in another frame.

There are many such puzzles which can only be resolved if the phenomena of length contraction and time dilation are taken into account. So consider two positive charges that move parallel to each other with speed v relative to the black frame (the lab); see Figure **A.9a**.

For an observer moving along with the charges (red frame) the charges are at rest and so there cannot be a magnetic force between them. There is, however, an electric force of repulsion. For this observer, the charges move away from each other along straight lines. For an observer at rest in the lab there are repulsive electric forces but there are also magnetic forces; this is because each moving charge creates a magnetic field and the other charge moves in that magnetic field. The magnetic field created by the bottom charge at the position of the top charge is directed out of the page. The magnetic force on the top charge is therefore directed opposite to the electric force; the magnetic force is attractive. The net force is still repulsive and the two charges move away from each other along curved paths. Examining the details of this situation shows that the two observers will reach consistent results only if time runs differently in the two different frames. This is more evidence that, to avoid these electromagnetic puzzles, ideas similar to Lorentz's must be true; it is not a surprise that Einstein's 1905 paper is entitled 'On the electrodynamics of moving bodies'.

Worked example

A.2 A positive electric charge q enters a region of magnetic field B with speed v (Figure **A.10**). Discuss the forces, if any, that the charge experiences according to **a** a frame of reference at rest with respect to the magnetic field (black) and **b** a frame moving with the same velocity as the charge (red).

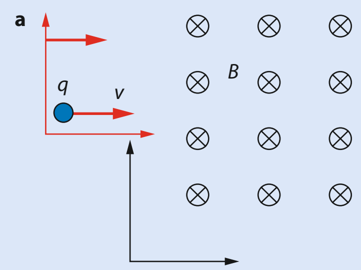


Figure A.10

a The charge will experience a magnetic force given by $F = qvB$.

b The charge is at rest. Hence the magnetic force is zero. But there has to be a force, and that can only be an electric force.

Nature of science

Paradigm shift

A fundamental fact of relativity theory is that the speed of light is constant for all inertial reference frames. This simple postulate has far-reaching consequences for our understanding of space and time. The idea that time is an absolute, assumed to be correct for over 2000 years, was shown to be false. Accepting the new ideas was not easy, but they offered the best explanation of observations and revolutionised physics.

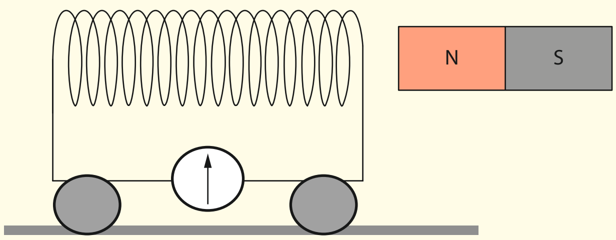
? Test yourself

1 It was a very hotly debated subject centuries ago as to whether the Earth goes around the Sun or the other way around. Does relativity make this whole argument irrelevant since 'all frames of reference are equivalent'?

2 Discuss the approximations necessary in order to claim that the rotating Earth is an inertial frame of reference.

3 Imagine that you are travelling in a train at constant speed in a straight line and that you cannot look at or communicate with the outside. Think of the first experiment that comes to your mind that you could do to try to find out that you are indeed moving. Then analyse it carefully to see that it will not work.

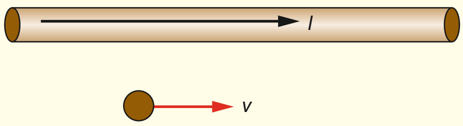
HL **4** Here is another experiment that could be performed in the hope of determining whether you are moving or not. The coil shown below is placed near a strong magnet, and a galvanometer attached to the coil registers a current. Discuss whether we can deduce that the coil moves with respect to the ground.



5 Outline an experiment you might perform in a train that is accelerating along a straight line that would convince you that it is accelerating. Discuss whether you could also determine the direction of the acceleration.

6 Outline how you would know that you find yourself in a rotating frame of reference.

7 An electric current flows in a wire. A proton moving parallel to the wire will experience a magnetic force due to the magnetic field created by the current. From the point of view of an observer travelling along with the proton, the proton is at rest and so should experience no force. Analyse the situation.



8 An inertial frame of reference S′ moving to the right with speed $15\,\mathrm{m\,s^{-1}}$ moves past another inertial frame S. At time zero the origins of the two frames coincide. **a** A phone rings at location $x = 20\,\mathrm{m}$ and $t = 5.0\,\mathrm{s}$. Determine the location of this event in the frame S′. **b** A ball is measured to have velocity $5.0\,\mathrm{m\,s^{-1}}$ in frame S′. Calculate the velocity of the ball in frame S.

9 An inertial frame of reference S′ moving to the left with speed $25\,\mathrm{m\,s^{-1}}$ moves past another inertial frame S. At time zero the origins of the two frames coincide. **a** A phone rings at location $x' = 24\,\mathrm{m}$ and $t = 5.0\,\mathrm{s}$. Determine the location of this event in the frame S. **b** A ball is measured to have velocity $-15\,\mathrm{m\,s^{-1}}$ in frame S. Calculate the velocity of the ball in frame S′.

Learning objectives

- Understand the two postulates of special relativity.
- Describe clock synchronisation.
- Understand and use the Lorentz transformations.
- Apply the velocity addition formula.
- Work with invariant quantities.
- Understand and apply time. dilation and length contraction.
- Use muon decay as evidence for relativity.

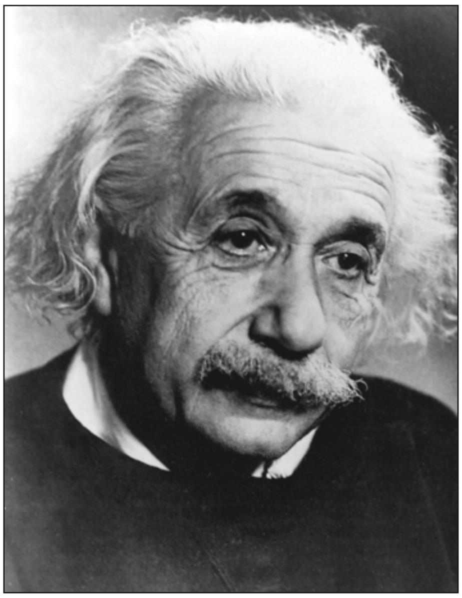


Figure A.11 Albert Einstein (1879–1955).

A2 The Lorentz transformations

This section introduces the postulates of the theory of relativity and the transformation equations that relate the coordinates of an event as seen from two different inertial frames. We discuss the two major consequences of these equations: time dilation and length contraction. We also discuss the **muon decay** experiment, which provides evidence in support of the theory of relativity.

A2.1 The postulates of special relativity

As discussed in Section **A1**, the Lorentz transformation equations are the simplest set of transformations with which the Maxwell theory is consistent when looked at from different reference frames. Einstein (Figure **A.11**) re-derived the same set of equations based on two much simpler and more general assumptions. These two assumptions are known as the **postulates of relativity**. They are:

> The laws of physics are the same in all inertial frames.
> The speed of light in vacuum is the same for all inertial observers.

So we see that it is not just electromagnetism that must be consistent with these transformations, but all the laws of physics.

These two postulates of relativity, although they sound simple, have far-reaching consequences. The fact that the speed of light in a vacuum is the same for all observers means that absolute time does not exist. Consider a beam of light. Two different observers in relative motion to each other will measure different distances covered by this beam. But if they are to agree that the speed of the beam is the same for both observers, it follows that they must also measure different times of travel. Thus, observers in motion relative to each other measure time differently. The constancy of the speed of light means that space and time are now inevitably linked and are not independent of each other, as they were in Newtonian mechanics.

The strength of intuition

Einstein was so convinced of the constancy of the speed of light that he elevated it to one of the principles of relativity. But it was not until 1964 that conclusive experimental verification of this took place. In an experiment at CERN, neutral pions moving at $0.99975c$ decayed into a pair of photons moving in different directions. The speed of the photons in both directions was measured to be c with extraordinary accuracy. The speed of light does not depend on the speed of its source.

We have seen that Galileo's transformation laws

$$x' = x - vt,\ t' = t,\ u' = u - v$$

need to be changed. Einstein (and Lorentz) modified them to

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}},\ t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Using the **gamma factor**, $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$, we may rewrite these as

$$x' = \gamma(x - vt),\ t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

These formulas are useful when we know x and t and we want to find x' and t'. If, on the other hand, we know x' and t' and want to find x and t, then

$$x = \gamma(x' + vt'),\ t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

These equations may be used to relate the coordinates of a **single** event in one inertial frame to those in another. We will denote by S some inertial frame. A frame that moves with speed v to the right relative to S we will call S′. These equations assume that, when clocks in both frames show zero (i.e. $t = t' = 0$), the origins of the two frames coincide (i.e. $x = x' = 0$).

Note that $\gamma > 1$. A graph of the gamma factor γ versus velocity is shown in Figure **A.12**. We see that γ is approximately 1 for velocities up to about half the speed of light, but approaches infinity as the speed approaches the speed of light.

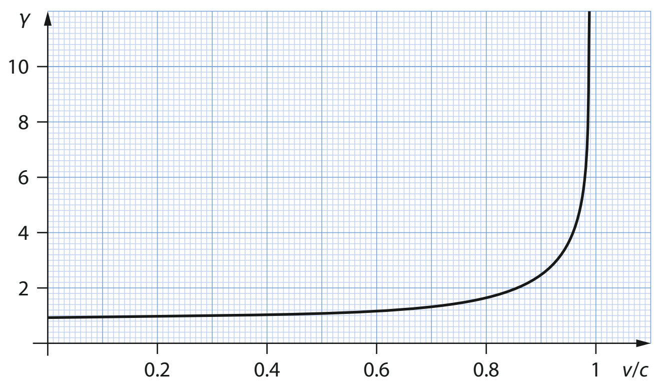


Figure A.12 The gamma factor γ as a function of velocity. The value of γ stays essentially close to 1 for values of the velocity up to about half the speed of light, but approaches infinity as the velocity approaches the speed of light.

A2.2 Clock synchronisation

The equations above also assume that the clocks in each frame are synchronised. This means that they show the same time at any given instant of time, i.e. we have **clock synchronisation**. How can this be achieved? The idea is to use the fact that the speed of light is a universal constant. Consider a clock that is a distance x from the origin. A light signal leaving the origin at time zero will take a time $\frac{x}{c}$ to arrive at the clock.

So we set that clock to show a time equal to $\frac{x}{c}$ and wait for a light signal from the origin to arrive. When the light signal does arrive, the clock is started (the clock is a stopwatch). Doing this for all the clocks in the reference frame ensures that they are all synchronised.

Worked examples

A.3 Lightning strikes a point on the ground (frame S) at position $x = 3500\,\text{m}$ and time $t = 5.0\,\text{s}$. Determine where and when the lightning struck according to a rocket that flies to the right over the ground at a speed of $0.80c$. (Assume that when clocks in both frames show zero, the origins of the two frames coincide.)

This is a straightforward application of the Lorentz formulas (the gamma factor is $\gamma = \frac{1}{\sqrt{1-0.80^2}} = \frac{5}{3}$):

$$\begin{aligned} x' &= \gamma(x - vt) \\ &= \frac{5}{3}(3500 - 0.80 \times 3 \times 10^8 \times 5.0) \\ &= -2.0 \times 10^9\,\text{m} \end{aligned}$$

and

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ &= \frac{5}{3}\left(5.0 - \frac{0.80 \times 3 \times 10^8 \times 3500}{(3 \times 10^8)^2}\right) \\ &= 8.3\,\text{s} \end{aligned}$$

The observers disagree on the coordinates of the event 'lightning strikes'; both are correct.

A.4 A clock in a rocket moving at $0.80c$ goes past a lab. When the origins of both frames coincide, all clocks are set to show zero. Calculate the reading of the rocket clock as it goes past the point with $x = 240\,\text{m}$.

The event 'clock goes past the given point' has coordinates x and t in the lab frame and x' and t' in the rocket frame. We need to find t'. We know that $x = 240\,\text{m}$. We calculate that $t = \frac{240}{0.80c} = 1.0 \times 10^{-6}\,\text{s}$. Hence, from the Lorentz formula for time.

$$\begin{aligned} t' &= \gamma\left(-\frac{vx}{c^2}\right) \\ &= \frac{5}{3}\left(1.0 \times 10^{-6} - \frac{0.80 \times 3 \times 10^8 \times 240}{(3 \times 10^8)^2}\right) \\ &= 0.60\,\mu\text{s} \end{aligned}$$

Exam tip
It is important that you are able to use the Lorentz equations to go from frame S to frame S′ as well as from S′ to S.

Sometimes we will be interested in the difference in coordinates of a pair of events. So, for events 1 and 2, we define

$$\Delta x' = x_2' - x_1' \text{ and } \Delta t' = t_2' - t_1' \text{ in S}'$$

and

$$\Delta x = x_2 - x_1 \text{ and } \Delta t = t_2 - t_1 \text{ in S}$$

These differences are related by

$$\Delta x' = \gamma(\Delta x - v\,\Delta t);\ \Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$$

or, in reverse,

$$\Delta x = \gamma(\Delta x' + v\,\Delta t');\ \Delta t = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right)$$

Worked examples

A.5 Consider a rocket that moves relative to a lab with speed $v = 0.80c$ to the right. We call the inertial frame of the lab S and that of the rocket S′. The length of the rocket as measured by an observer on the rocket is 630 m. A photon is emitted from the back of the rocket towards the front end. Calculate the time taken for the photon to reach the front end of the rocket according to observers in the rocket and in the lab.

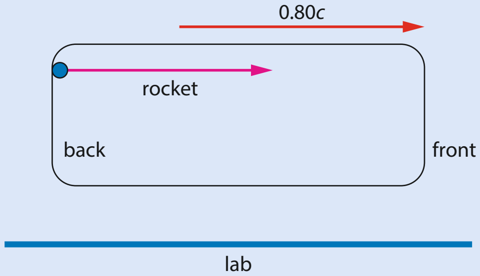


Figure A.13

Event 1 is the emission of the photon. Event 2 is the arrival of the photon at the front of the rocket. Clearly,

$$\Delta x' = x'_{\text{arrival}} - x'_{\text{emission}} = 630\,\text{m}$$

This distance is covered at the speed of light and so the time between emission and arrival is

$$\Delta t' = \frac{630}{c} = 2.1 \times 10^{-6}\,\text{s}$$

In frame S we use

$$\Delta x = \frac{5}{3}(630 + 0.80c \times 2.1 \times 10^{-6}) = 1890\,\text{m}$$

$$\Delta t = \frac{5}{3}\left(2.1 \times 10^{-6} + \frac{0.80c \times 630}{c^2}\right) = 6.30 \times 10^{-6}\,\text{s}$$

A.6 a A rocket approaches a space station with speed $0.980c$ (relative to the space station). Observers in the rocket record the firing of a missile from the space station and 1.0 s later (according to rocket clocks) record an explosion at another space station behind the rocket at a distance of 4.0×10^8 m (also measured according to the rocket). Calculate the distance and the time interval between the events 'firing of the missile' and 'explosion' according to observers in the space station. Comment on your answers.

b In frame S, event A *causes* event B and therefore occurs before event B. Show that in any frame, event A always occurs before event B.

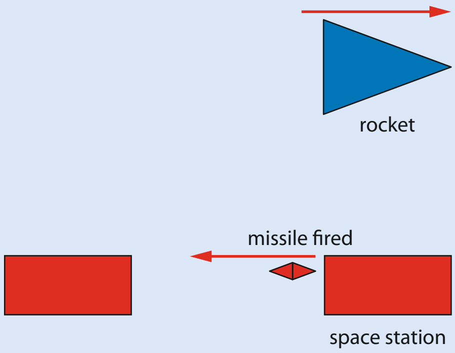


Figure A.14

a The gamma factor for a speed of $0.980c$ is $\gamma = \frac{1}{\sqrt{1-0.980^2}} \approx 5.0$. We will take the space station as frame S and the rocket as frame S′. Then we know that $\Delta x' = x'_{\text{expl}} - x'_{\text{fire}} = -4.0 \times 10^8$ m; $\Delta t' = t'_{\text{expl}} - t'_{\text{fire}} = 1.0$ s. The explosion happens after the firing of the missile and so $\Delta t' > 0$. We apply the (reverse) Lorentz transformations to find

$$\Delta x = 5.0 \times (-4.0 \times 10^8 + 0.98c \times 1.0) = -5.3 \times 10^8\ \text{m}$$

$$\Delta t = 5.0 \times \left(1.0 + \frac{0.98c \times (-4.0 \times 10^8)}{c^2}\right) = -1.5\ \text{s}$$

The negative answer for the time interval means that the explosion happens **before** the firing of the missile. Therefore the missile could not have been responsible for the explosion! We could have predicted that the missile had nothing to do with the explosion because, according to observers in the rocket, the missile would have to cover a distance of 4.0×10^8 m in 1.0 s, that is, with a speed of $4.0 \times 10^8\ \text{m s}^{-1}$, which exceeds the speed of light and so is impossible.

b We are told that $\Delta t = t_B - t_A > 0$. Let Δx be the distance between the two events. The fastest way information from A can reach B is at the speed of light, and so $\frac{\Delta x}{\Delta t} < c$. In any other frame,

$$\begin{aligned}
\Delta t' &= t'_B - t'_A \\
&= \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \\
&= \gamma \Delta t\left(1 - \frac{v\Delta x}{c^2 \Delta t}\right) \\
&> \gamma \Delta t\left(1 - \frac{vc}{c^2}\right) = \gamma \Delta t\left(1 - \frac{v}{c}\right) \\
&> 0
\end{aligned}$$

So event A has to occur before event B in any frame.

A2.3 Time dilation

Suppose that an observer in frame S′ measures the time between successive ticks of a clock. The clock is at rest in S′ and so the measurement of the ticks occurs at the same place, $\Delta x' = 0$. The result of his measurement is $\Delta t'$. The observer in S will measure a time interval equal to

$$\Delta t = \gamma\left(\Delta t' - \frac{v\,\Delta x'}{c^2}\right)$$

$$= \gamma(\Delta t' + 0)$$

$$\Delta t = \gamma\,\Delta t'$$

This shows that the interval between the ticks of a clock that is moving relative to the observer in S is greater than the interval measured in S′ where the clock is at rest. This is known as **time dilation**. In this case, the time measured in S′ is special because it represents a time interval between two events that happened at the same point in space. The clock is at rest in S′ so its ticks happen at the same point in S′. Such a time interval is called **proper time interval**. (This is just a name; it is not implied that this is a more 'correct' measurement of time.)

A proper time interval is the time in between two events that take place at the same point in space.

The **time dilation** formula is best remembered as

$$\text{time interval} = \gamma \times \text{proper time interval}$$

$$= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \times \text{proper time interval}$$

Exam tip
Do not make the mistake of thinking that proper time intervals are measured only in the frame S′. A proper time interval is the time between two events that take place at the same point.

Different observers disagree in their measurements, but both are right

Note that there is no question as to which observer is right and which is wrong when it comes to measuring time intervals. Both are right. Two inertial observers moving relative to each other at constant velocity both reach valid conclusions, according to the principle of relativity.

Worked examples

A.7 The time interval between the ticks of a clock carried on a fast rocket is half of what observers on the Earth record. Calculate the speed of the rocket relative to the Earth.

From the time dilation formula it follows that

$$2 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Rightarrow 1 - \frac{v^2}{c^2} = \frac{1}{4}$$

$$\Rightarrow \frac{v^2}{c^2} = \frac{3}{4}$$

$$\Rightarrow v = 0.866c$$

A.8 A rocket moves past an observer in a laboratory with speed $v = 0.85c$. An observer in the laboratory measures that a radioactive sample of mass 50 mg (which is at rest in the laboratory) has a half-life of 2.0 min. Calculate the half-life as measured by observers in the rocket.

We have two events here. The first is that the laboratory observer sees a container with 50 mg of the radioactive sample. The second event is that the laboratory observer sees a container with 25 mg of the radioactive sample. These events are separated by 2.0 min as far as the laboratory observer is concerned. These two events take place at the same point in space as far as the laboratory observer is concerned, so the laboratory observer has measured the **proper time interval** between these two events. Hence the rocket observers will measure a longer half-life, equal to $\gamma \times$ (proper time interval)

$$= \frac{1}{\sqrt{1 - 0.85^2}} \times 2.0\,\text{min}$$

$$= 3.80\,\text{min}$$

The point of this example is that you must not make the mistake of thinking that proper time intervals are measured by 'the moving' observer. There is no such thing as 'the moving' observer: the rocket observer is free to consider herself at rest and the laboratory observer to be moving with velocity $v = -0.85c$.

A2.4 Length contraction

Exam tip
Note that it is only lengths in the direction of motion that contract.

Now consider a rod that is at rest in frame S′. An observer in S′ measures the position of the ends of the rod, subtracts and finds a length $\Delta x' = L_0$. (Notice that, since the rod is at rest, the measurements of the position of the ends can be done at different times: the rod is not going anywhere). But for an observer in S the rod is moving, so to measure its length he must record the position of the ends of the rod at the same time, that is, with $\Delta t = 0$. Since $\Delta x' = \gamma(\Delta x - v\Delta t)$, we obtain the result that

$$L_0 = \gamma(L - 0)$$

$$L = \frac{L_0}{\gamma}$$

This shows that the rod, which is moving relative to the observer in S, has a shorter length in S than the length measured in S′, where the rod is at rest. This is known as **length contraction**.

The length $\Delta x' = L_0$ is special because it is measured in a frame of reference where the rod is at rest. Such a length is called a **proper length**.

The length of an object measured by an inertial observer with respect to whom the object is at rest is called a proper length. Observers with respect to whom the object moves at speed v measure a shorter length:

$$\text{length} = \frac{\text{proper length}}{\gamma}$$

We must accept experimentally verified observations, no matter how 'strange' they may appear

Time dilation as described is a 'real' effect. In the Hafele–Keating experiment, accurate atomic clocks taken for a ride aboard planes moving at ordinary speeds and then compared with similar clocks left behind show readings that are smaller by amounts consistent with the formulas of relativity. Time dilation is also a daily effect in the operation of particle accelerators. In such machines, particles are accelerated to speeds that are very close to the speed of light and thus relativistic effects must be taken into account when designing these machines. The time dilation formula has also been verified in muon- decay experiments.

Worked example

A.9 In the year 2014, a group of astronauts embark on a journey towards the star Betelgeuse in a spacecraft moving at $v = 0.75c$ with respect to the Earth. Three years after departure from the Earth (as measured by the astronauts' clocks) one of the astronauts announces that she has given birth to a baby girl. The other astronauts immediately send a radio signal to the Earth announcing the birth. Calculate when the good news is received on the Earth (according to Earth clocks).

When the astronaut gives birth, three years have gone by according to the spacecraft's clocks. This is a proper time interval since the events 'departure from Earth' and 'astronaut gives birth' happen at the same place as far as the astronauts are concerned (inside the spacecraft). Thus, the time between these two events according to the Earth clocks is

$$\text{time interval} = \gamma \times \text{proper time interval}$$

$$= \frac{1}{\sqrt{1-0.75^2}} \times 3.0\,\text{yr}$$

$$= 4.54\,\text{yr}$$

This is therefore also the time during which the spacecraft has been travelling, as far as the Earth is concerned. The distance covered is (as far as the Earth is concerned)

$$\text{distance} = vt$$

$$= 0.75c \times 4.54\,\text{yr}$$

$$= 3.40c\ \text{yr}$$

$$= 3.40\,\text{ly}$$

Exam tip
1 ly = c × 1 year

This is the distance that the radio signal must then cover in bringing the message. This is done at the speed of light and so the time taken is

$$\frac{340\,\text{ly}}{c} = \frac{3.40c \times \text{yr}}{c}$$

$$= 3.40\,\text{yr}$$

Hence, when the signal arrives, the year on the Earth is $2014 + 4.54 + 3.40 = 2022$ (approximately).

A2.5 Another look at time dilation

We have seen using Lorentz transformations that relativity predicts the phenomenon of time dilation. A simpler view of this effect is the following. A direct consequence of the principle of relativity is that observers who are in motion relative to each other do not agree on the interval of time separating two events. To see this, consider the following situation: a train moves with velocity v with respect to the ground as shown in Figure **A.15**.

From point A on the train floor, a light signal is sent towards point B directly above on the ceiling. The time $\Delta t'$ it takes for light to travel from A to B and back to A is recorded. Note that as far as the observers inside the train are concerned, the light beam travels along the straight-line segments AB and BA. From the point of view of an observer on the ground, however, things look somewhat different. In the time it takes for light to return to A, point B (which moves along with the train) will have moved forward. This means, therefore, that to this observer the path of the light beam looks like that shown in Figure **A.16**.

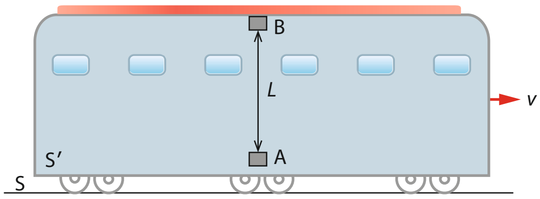


Figure A.15 A signal is emitted at A, is reflected off B and returns to A again. The path shown is what the observer on the train sees as the path of the signal. The emission and reception of the signals happen at the same point in space.

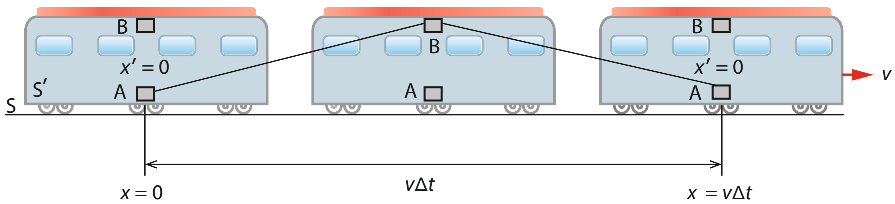


Figure A.16 The ground observer sees things differently. In the time it takes the signal to return, the train has moved forward. Thus, the emission and reception of the signal do not happen at the same point in space. (The diagram is exaggerated for clarity.)

Suppose that the stationary observer measures the time it takes light to travel from A to B and back to A to be Δt. What we will show now is that, if both observers are to agree that the speed of light is the same, then the two time intervals cannot be the same.

It is clear that

$$\Delta t' = \frac{2L}{c}$$

$$\Delta t = \frac{2\sqrt{L^2 + (v\Delta t/2)^2}}{c}$$

since the train moves forward a distance $v\Delta t$ in the time interval Δt that the stationary observer measures for the light beam to reach A. Thus, solving the first equation for L and substituting in the second (after squaring both equations)

$$(\Delta t)^2 = \frac{4\left(\frac{c^2(\Delta t')^2}{4} + \frac{v^2(\Delta t)^2}{4}\right)}{c^2}$$

$$c^2(\Delta t)^2 = c^2(\Delta t')^2 + v^2(\Delta t)^2$$

$$(c^2 - v^2)(\Delta t)^2 = c^2(\Delta t')^2$$

$$(\Delta t)^2 = \frac{c^2(\Delta t')^2}{c^2 - v^2}$$

So, finally,

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This is the **time dilation** formula. It is customary to call the expression

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

the gamma factor, in which case $\Delta t = \gamma \Delta t'$. Recall that $\gamma > 1$.

The time interval for the travel of the light beam is longer on the ground observer's clock. This is known as time dilation. If the train passengers measure a time interval of $\Delta t' = 6.0\,\mathrm{s}$ and the train moves at a speed $v = 0.80c$, then the time interval measured by the ground observer is

$$\Delta t = \frac{6.0}{\sqrt{1 - 0.80^2}}\,\mathrm{s}$$

$$= \frac{6.0}{\sqrt{1 - 0.64}}\,\mathrm{s}$$

$$= \frac{6.0}{\sqrt{0.36}}\,\mathrm{s}$$

$$= \frac{6.0}{0.6}\,\mathrm{s}$$

$$= 10\,\mathrm{s}$$

which is longer. It will be seen immediately that this large difference came about only because we chose the speed of the train to be extremely close to the speed of light. Clearly, if the train speed is small compared with the speed of light, then $\gamma \approx 1$, and the two time intervals agree, as we might expect them to from everyday experience. The reason that our everyday experience leads us astray is because the speed of light is enormous compared with everyday speeds. Thus, the relativistic time dilation effect we have just discovered becomes relevant only when speeds close to the speed of light are encountered.

A2.6 Addition of velocities

Consider a frame S′ (for example, a train) that moves at constant speed v in a straight line relative to another frame S (for example, the ground). An object slides on the train floor in the same direction as the train (S′) and its velocity is measured, by observers in S′, to be u'. What is the speed u of this object as measured by observers in S? (See Figure **A.17**.)

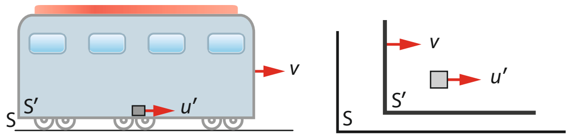


Figure A.17 The speed of the moving object is u' in the frame S′. What is its speed as measured in frame S?

In pre-relativity physics (i.e. Galilean–Newtonian physics), the answer would be simply $u'+v$. This cannot, however, be the correct relativistic answer; if we replaced the sliding object by a beam of light ($u'=c$), we would end up with an observer (S) who measured a speed of light different from that measured by S′. The correct answer for the speed u of the particle relative to S is

$$u=\frac{u'+v}{1+\dfrac{u'v}{c^2}}$$

or, solving for u',

$$u'=\frac{u-v}{1-\dfrac{uv}{c^2}}$$

It can easily be checked that, irrespective of how close u' or v is to the speed of light, u is always less than c. For the case in which $u'=c$, then $u=c$ as well, as demanded by the principle of the constancy of the speed of light (check this). On the other hand, if the velocities involved are small compared with the speed of light, then we may neglect the term $\frac{u'v}{c^2}$ in the denominator, in which case Einstein's formula reduces to the familiar Galilean relativity formula $u=u'+v$.

Worked examples

A.10 An electron has a speed of $2.00\times10^8\,\mathrm{m\,s^{-1}}$ relative to a rocket, which itself moves at a speed of $1.00\times10^8\,\mathrm{m\,s^{-1}}$ with respect to the ground. What is the speed of the electron with respect to the ground?

Applying the formula above with $u'=2.00\times10^8\,\mathrm{m\,s^{-1}}$ and $v=1.00\times10^8\,\mathrm{m\,s^{-1}}$, we find $u=2.45\times10^8\,\mathrm{m\,s^{-1}}$.

A.11 Two rockets move away from each other with speeds of $0.8c$ and $0.9c$ with respect to the ground, as shown in Figure **A.18**. What is the speed of each rocket as measured from the other? What is the relative speed of the two rockets as measured from the ground?

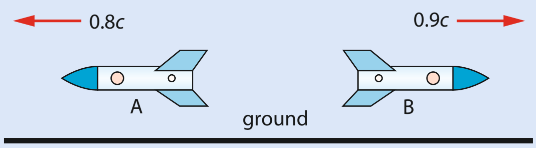


Figure A.18

Let us first find the speed of A with respect to B. In the frame of reference in which B is at rest, the ground moves to the left with speed $0.90c$ (i.e. a velocity of $-0.90c$). The velocity of A with respect to the ground is $-0.80c$. This is illustrated in Figure **A.19**.

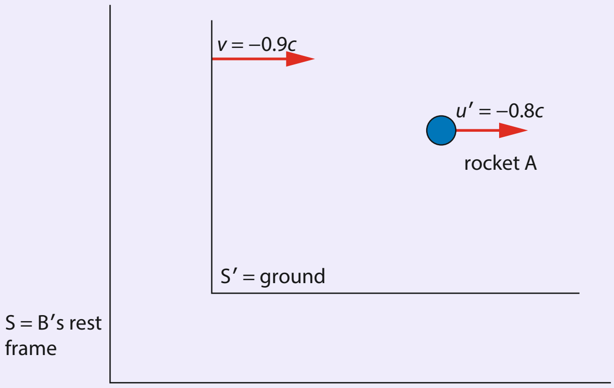


Figure A.19

Applying the formula with the values given in the figure, we find

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

$$= \frac{-0.90c - 0.80c}{1 + \frac{(-0.90c)(-0.80c)}{c^2}}$$

$$= \frac{-1.70c}{1 + 0.72}$$

$$= -\frac{1.70c}{1.72}$$

$$= -0.988c \approx -0.99c$$

The minus sign means, of course, that rocket A moves to the left relative to B. Let us now find the speed of rocket B as measured in A's **rest frame**. The appropriate diagram is shown in Figure **A.20**.

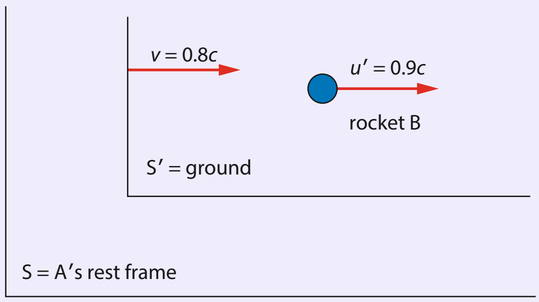


Figure A.20

We thus find

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

$$= \frac{0.90c + 0.80c}{1 + \frac{(0.90c)(0.80c)}{c^2}}$$

$$= \frac{1.70c}{1 + 0.72}$$

$$= \frac{1.70c}{1.72}$$

$$= 0.988c \approx 0.99c \text{ as expected.}$$

A.12 A rocket has a proper length of 250 m and travels at a speed $v = 0.950c$ relative to the Earth. A missile is fired from the back of the rocket at a speed $u' = 0.900c$ relative to the rocket. **a** Calculate the time when the missile passes the front of the rocket according to **i** observers in the rocket and **ii** according to observers on the Earth. **b** Calculate the speed of the missile relative to the Earth and show that your answer is consistent with the answer in **a**.

a **i** Calling, as usual, the Earth frame S and the rocket frame S′, we know that $\Delta x' = 250$ m and

$\Delta t' = \frac{250}{0.900 \times 3.0 \times 10^8} = 9.259 \times 10^{-7} \approx 9.26 \times 10^{-7}$ s, where we are considering the events 'missile is fired' and 'missile passes the front of the rocket'.

ii We need to find Δt. From the Lorentz formulas we know that $\Delta t = \gamma\left(\Delta t' + \frac{v\Delta x}{c^2}\right)$. The gamma factor is $\gamma = \frac{1}{\sqrt{1 - 0.950^2}} = 3.20$. Hence

$$\Delta t = 3.20 \times \left(9.259 \times 10^{-7} + \frac{0.950 \times 250}{3.0 \times 10^8}\right)$$

$$= 5.496 \times 10^{-6} \approx 5.50 \times 10^{-6}\,\text{s}$$

b From the relativistic addition law for velocities,

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

$$= \frac{0.900c + 0.950c}{1 + \frac{(0.900c)(0.950c)}{c^2}}$$

$$= 0.997c$$

The distance travelled by the missile according to Earth observers is

$$\Delta x = \gamma(\Delta x' + v\Delta t')$$

$$= 3.20 \times (250 + 0.950 \times 3.0 \times 10^8 \times 9.26 \times 10^{-7})$$

$$= 1645\,\text{m}$$

This distance was covered at a speed of $0.997c$ and so the time taken, according to Earth observers, is $\frac{1645}{0.997 \times 3.0 \times 10^8} = 5.50 \times 10^{-6}$ s, as expected from **a**.

A2.7 Simultaneity

Another great change introduced into physics as a result of relativity is the concept of **simultaneity**. Imagine three rockets A, B and C travelling with the same constant velocity (with respect to some inertial observer) along the same straight line. Imagine that rocket B is halfway between rockets A and C, as shown in Figure **A.21**.

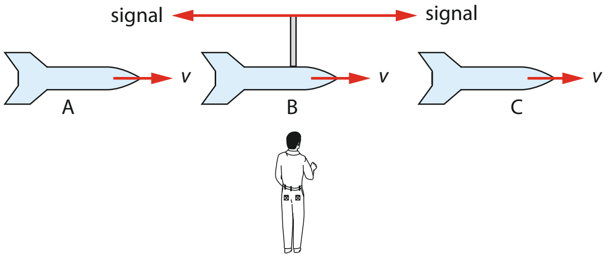


Figure A.21 B emits signals to A and C. These are received at the same time as far as B is concerned. But the reception of the signals is not simultaneous as far as the ground observer is concerned.

Rocket B emits light signals that are directed towards rockets A and C. Which rocket will receive the signal first? The principle of relativity allows us to determine that, as far as an observer in rocket B is concerned, the signals are received by A and C simultaneously (i.e. at the same time). This is obvious, since we may imagine a big box enclosing all three rockets that moves with the same velocity as the rockets themselves. Then, for any observer in the box, or in the rockets themselves, everything appears to be at rest. Since B is halfway between A and C, clearly the two rockets receive the signals at the same time (light travels with the same speed). Imagine, though, that we look at this situation from the point of view of a different observer, outside the rockets and the box, with respect to whom the rockets move with velocity v. This observer sees that rocket A is approaching the light signal, while rocket C is moving away from it (again, remember that light travels with the same speed in each direction). Hence, it is obvious to this observer that rocket A will receive the signal **before** rocket C does.

In the next section we will use the Lorentz equations to prove that:

Events that are simultaneous for one observer and which take place at **different points in space** are not simultaneous for another observer in motion relative to the first.

On the other hand, if two events are simultaneous for one observer and take place at **the same point in space**, they are simultaneous for all other observers as well.

Worked example

A.13 Observer T is in the middle of a train that is moving with constant speed to the right with respect to the train station. Two light signals are emitted at the same time as far as the observer T in the train is concerned (Figure **A.22**).

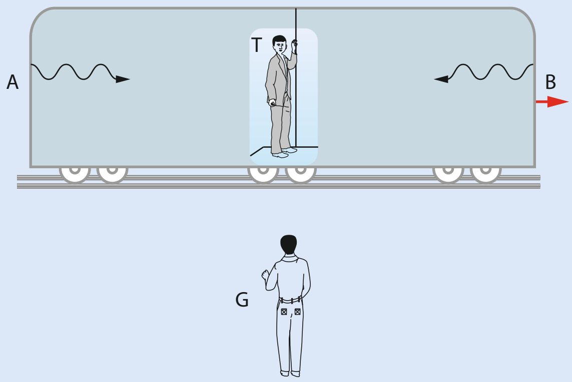


Figure A.22

a Determine whether the emissions are simultaneous for an observer G on the ground.

b The signals arrive at T at the same time as far as T is concerned.

i Determine whether they arrive at T at the same time as far as G is concerned.

ii Deduce, according to G, which signal is emitted first.

a No, because the events (i.e. the emissions from A and from B) take place at different points in space, and so if they are simultaneous for observer T they will not be simultaneous for observer G.

b **i** Yes, because the reception of the two signals by T takes place at the same point in space, so if they are simultaneous for T, they must also be simultaneous for G.

ii From G's point of view, T is moving away from the signal from A. So the signal from A has a larger distance to cover to get to T. If the signals are received at the same time, and moved at the same speed c, it must be that the one from A was emitted before that from B.

Simultaneity, like motion, is a relative concept. Our notion of absolute simultaneity is based on the idea of absolute time: events happen at specific times that all observers agree on. Einstein has taught us that the idea of absolute time, just like the idea of absolute motion, must be abandoned.

The Lorentz equations allow for a more quantitative approach to simultaneity. Suppose that as usual we have frames S and S′, with frame S′ moving to the right with speed v relative to S. Suppose that two events take place at the same time in frame S. The time interval between these two events is thus zero: $\Delta t = 0$. What is the time interval between the same two events when measured in S′? The Lorentz equations give

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$$

$$= -\gamma\frac{v}{c^2}\Delta x$$

We have the interesting observation that, if $\Delta x \neq 0$, i.e. if the simultaneous events in S occur at the same point in space, then they are also simultaneous in all other frames. However, if $\Delta x \neq 0$, then the events will not be simultaneous in other frames.

Worked examples

A.14 Let us rework the previous example quantitatively. Take the proper length of the train (frame S′) to be 300 m and let it move with speed $0.98c$ to the right. Determine, according to an observer on the ground (frame S), which light turns on first and by how much.

We are told that $\Delta t' = 0$ and $\Delta x' = x'_B - x'_A = 300$ m. We want to find Δt, the time interval between the events representing the emission of light from A and from B. The gamma factor is $\gamma = \frac{1}{\sqrt{1-0.98^2}} = 5.0$. Then

$$\Delta t = t_B - t_A$$

$$= \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right)$$

$$= 5.0 \times \left(0 + \frac{0.98c}{c^2} \times 300\right) = +4.9 \times 10^{-6}\,\text{s}$$

The positive sign indicates that $t_B > t_A$, that is, the light from A was emitted 4.9 μs before the light from B.

Note: if we wanted to find the actual emission times of the lights and not just their difference, we would use

$$t_A = \gamma\left(t' + \frac{v}{c^2}x'\right) = 5.0 \times \left(0 + \frac{0.98c}{c^2} \times (-150)\right) = -2.45 \times 10^{-6}\,\text{s}$$

and

$$t_B = \gamma\left(t' + \frac{v}{c^2}x'\right) = 5.0 \times \left(0 + \frac{0.98c}{c^2} \times (+150)\right) = +2.45 \times 10^{-6}\,\text{s}$$

giving the same answer for the difference.

A.15 Let us look at the previous example again, but now we will assume that the emissions from A and B happened at the same time for observer G on the ground, in frame S (Figure **A.23**). Determine which emission takes place first in the train frame, frame S′.

We are now told that $\Delta t = 0$. In frame S, light A is emitted from position $x_A = -a$ and light B from position $x_B = +a$.

We want to find $\Delta t'$. The gamma factor is $\gamma = \frac{1}{\sqrt{1-0.98^2}} = 5.0$. Then

$$\Delta t' = t'_B - t'_A$$

$$= \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$$

$$= 5.0 \times \left(0 - \frac{0.98c}{c^2} \times 2a\right) = -3.3 \times 10^{-8}\,a$$

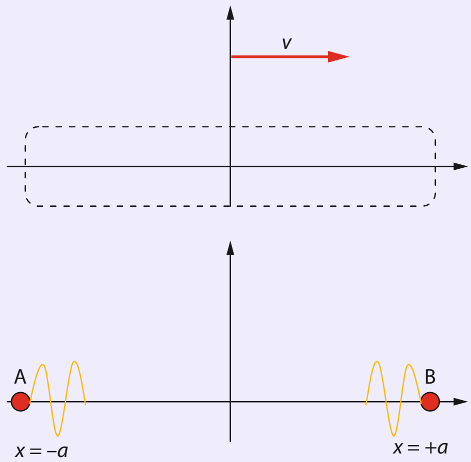


Figure A.23

The negative sign shows that according to observer T, light B was emitted first. But to get a numerical answer we need to know a. The light signals are emitted from the ends of the train, of proper length 300 m. In the frame S this length is Lorentz-contracted to 60 m, and so $a = 30$ m. (You can also obtain this result as follows. In the frame S, light B is emitted at $t = 0$. The position of this event in S′ is $x' = 150$ m.

Using $x' = \gamma(x - vt)$, $150 = 5.0(a - 0) \Rightarrow a = \frac{150}{5.0} = 30$ m.)

Hence, $\Delta t' = -3.3 \times 10^{-8} \times 30 = -9.9 \times 10^{-7}$ s.

A2.8 Muon decay

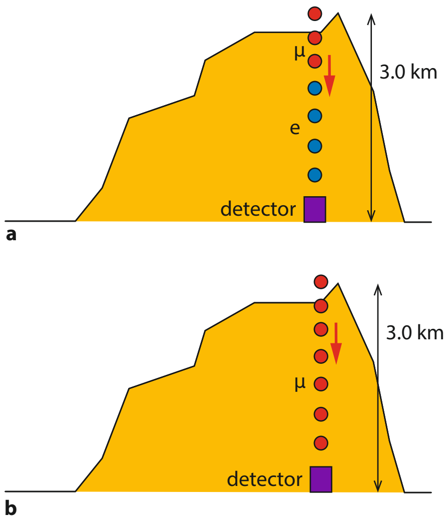


Figure A.24 **a** If relativity did not hold, muons created at the top of the mountain would not have enough time to reach the bottom as muons. **b** Because of relativistic effects, muons do reach the bottom of the mountain.

Muons are particles with properties similar to those of electrons except that they are more massive, are unstable and decay to electrons; they have an average lifetime of about 2.2×10^{-6} s. (The reaction is $\mu^- \rightarrow e^- + \overline{\nu}_e + \nu_\mu$.) This is the lifetime measured in a frame where the muon is at rest: the proper time interval between the creation of a muon and its subsequent decay.

From the point of view of an observer in the laboratory, however, the muons are moving at high speed, and the lifetime is longer because of time dilation. Consider a muon created by a source at the top of a mountain 3.0 km tall (Figure **A.24**). The muon travels at $0.99c$ towards the surface of the Earth.

Without relativistic time dilation, the muon would have travelled a distance (as measured by ground observers) of only

$$0.99 \times 3 \times 10^8 \times 2.2 \times 10^{-6}\,\mathrm{m} = 0.653\,\mathrm{km}$$

before decaying to an electron (Figure **A.24a**). Thus a detector at the base of the mountain would record the arrival of an electron, not a muon.

But experiments show the arrival of muons at the detector. This is because the lifetime of the muon as measured by ground observers is

$$\begin{aligned}\text{time interval} &= \frac{\text{proper time}}{\sqrt{1-\frac{v^2}{c^2}}} \\ &= \frac{2.2 \times 10^{-6}\,\mathrm{s}}{\sqrt{1-0.99^2}} \\ &= 1.56 \times 10^{-5}\,\mathrm{s}\end{aligned}$$

In this time the muon travels a distance (as measured by ground observers) of $0.99 \times 3 \times 10^8 \times 1.56 \times 10^{-5}\,\mathrm{m} = 4.63\,\mathrm{km}$

This means that the muon reaches the surface of the Earth before decaying (Figure **A.24b**).

> The fact that muons do make it to the surface of the Earth is evidence in support of the time dilation effect.

The muon exists as a muon for only 2.2×10^{-6} s in the muon's rest frame. So how does an observer travelling along with the muon explain the arrival of muons (and not electrons) at the surface of the Earth (Figure **A.25**)?

The answer is that the distance of 3.0 km measured by observers on the Earth is a proper length for them but not for the observer at rest with respect to the muon. This observer claims that it is the Earth that is moving upwards, and so measures a length-contracted distance of

$$3.0 \times \sqrt{1-0.99^2}\,\mathrm{km} = 0.42\,\mathrm{km}$$

to the surface of the Earth. The Earth's surface is coming up to this observer with a speed of $0.99c$ and so the time when they will meet is

$$\frac{0.42 \times 10^3}{0.99 \times 3 \times 10^8}\,\mathrm{s} = 1.4 \times 10^{-6}\,\mathrm{s}$$

that is, before the muon decays.

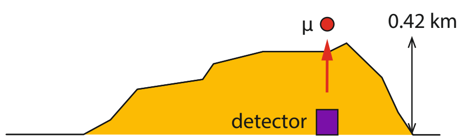


Figure A.25 From the point of view of an observer on the muon, the mountain is much shorter.

> In this sense, muon decay experiments are indirect confirmations of the length contraction effect.

Worked example

A.16 At the Stanford Linear Accelerator, electrons of speed $v = 0.960c$ move a distance of 3.00 km.

a Calculate how long this takes according to observers in the laboratory.

b Calculate how long this takes according to an observer travelling along with the electrons.

c Find the speed of the linear accelerator in the rest frame of the electrons.

a In the laboratory, the electrons take a time of

$$\frac{3.00 \times 10^3}{0.960 \times 3.00 \times 10^8}\,\text{s} = 1.04 \times 10^{-5}\,\text{s}$$

b The arrival of the electrons at the beginning of the accelerator track and that the end happen at the same point in space as far as the observer travelling with the electrons is concerned, so this is a proper time interval. Thus

$$\text{time interval} = \gamma \times \text{proper time interval}$$

$$\gamma = \frac{1}{\sqrt{1 - 0.960^2}}$$

$$= 3.571$$

$$1.0420 \times 10^{-5}\,\text{s} = 3.571 \times \text{proper time interval}$$

That is,

$$\text{proper time interval} = \frac{1.04 \times 10^{-5}}{3.571}$$

$$= 2.91 \times 10^{-6}\,\text{s}$$

c The speed of the accelerator is obviously $v = 0.960c$ in the opposite direction. But this can be checked as follows. As far as the electron is concerned, the length of the accelerator track is moving past it and so is length-contracted according to

$$\text{length} = \frac{\text{proper length}}{\gamma}$$

$$= \frac{3.00\,\text{km}}{3.571}$$

$$= 0.840\,\text{km}$$

and so has a speed of

$$\text{speed} = \frac{0.840 \times 10^3}{2.91 \times 10^{-6}}\,\text{m s}^{-1}$$

$$= 2.89 \times 10^8\,\text{m s}^{-1}$$

$$\approx 0.96\,c$$

Nature of science

Pure science – applying general arguments to special cases

Some years before relativity was introduced by Einstein, an experiment performed by Albert Michelson and Edward Morley gave a very puzzling result concerning the speed of light. The experimenters had expected to find that light travelled faster when it was directed along the path of travel of the Earth than when it was at right angles to the Earth's motion. They found no difference. There were frantic attempts to resolve the difficulties posed by this experiment. One attempt was to assume that moving lengths appear shorter. Lorentz showed that if one used his Lorentz transformation equations, certain difficulties with electromagnetism went away. But it was Einstein who re-derived these transformation equations from far more general principles, the postulates of relativity. By demanding that all inertial observers experience the same laws of physics and measured the same velocity of light in vacuum, the Lorentz equations emerged as the simplest linear equations that could achieve this.

? Test yourself

10 An earthling sits on a bench in a park eating a sandwich. It takes him 5 min to finish it, according to his watch. He is being monitored by invaders from planet Zenga who are orbiting at a speed of $0.90c$.

a Calculate how long the aliens reckon it takes an earthling to eat a sandwich.

b The aliens in the spacecraft get hungry and start eating their sandwiches. It takes a Zengan 5 min to eat her sandwich according to Zengan clocks. They are actually being observed by earthlings as they fly over the Earth. Calculate how long it takes a Zengan to eat a sandwich according to Earth clocks.

11 A cube has density ρ when the density is measured at rest. Suggest what happens to the density of the cube when it travels past you at a relativistic speed.

12 A pendulum in a fast train is found by observers on the train to have a period of 1.0 s. Calculate the period that observers on a station platform would measure as the train moves past them at a speed of $0.95c$.

13 A spacecraft moves past you at a speed of $0.95c$ and you measure its length to be 100 m. Calculate the length you would measure if it were at rest with respect to you.

14 Two identical fast trains move parallel to each other. An observer on train A tells an observer on train B that by her measurements (i.e. by A's measurements) train A is 30 m long and train B is 28 m long.
The observer on train B takes measurements; calculate what he will he find for:

a the speed of train A with respect to train B

b the length of train A

c the length of train B.

15 An unstable particle has a lifetime of 5.0×10^{-8} s as measured in its rest frame. The particle is moving in a laboratory with a speed of $0.95c$ with respect to the lab.

a Calculate the lifetime of the particle according to an observer at rest in the laboratory.

b Calculate the distance travelled by the particle before it decays, according to the observer in the laboratory.

16 The star Vega is about 50 ly away from the Earth. A spacecraft moving at $0.995c$ is heading towards Vega.

a Calculate how long it will take the spacecraft to get to Vega according to clocks on the Earth.

b The crew of the spacecraft consists of 18-year-old IB graduates. Calculate how old the graduates will be (according to their clocks) when they arrive at Vega.

17 A rocket travelling at $0.60c$ with respect to the Earth is launched towards a star. After 4.0 yr of travel (as measured by clocks on the rocket) a radio message is sent to the Earth. Calculate when it will arrive on the Earth as measured by:

a observers on the Earth

b observers on the rocket.

18 A spacecraft leaving the Earth with a speed of $0.80c$ sends a radio signal to the Earth as it passes a space station 8.0 ly from the Earth (as measured from the Earth).

a Calculate how long it takes the signal to arrive on the Earth according to Earth observers.

b Calculate how long it takes the spacecraft to reach the space station (according to clocks on the spacecraft).

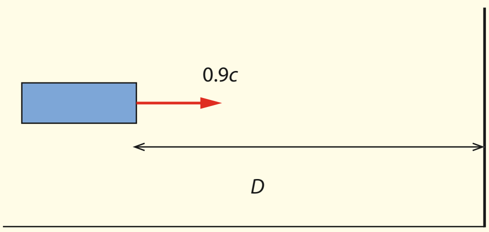
c As soon as the signal is received on the Earth, a reply signal is sent to the spacecraft. Calculate how long the reply signal takes to arrive at the spacecraft, according to Earth clocks.

d According to spacecraft clocks, calculate how much time goes by between the emission of the signal and the arrival of the reply.

19 A rocket approaches a mirror on the ground at a speed of $0.90c$, as shown below. The distance D between the front of the rocket and the mirror is 2.4×10^{12} m, as measured by observers on the ground, when a light signal is sent towards the mirror from the front of the rocket. Calculate when the reflected signal is received by the rocket as measured by:

a observers on the ground

b observers on the rocket.



20 Two objects move along the same straight line. Their speeds are as measured by an observer on the ground. Find:

a the velocity of B as measured by A

b the velocity of A as measured by B.

21 Repeat Question **20** for the arrangement below.

22 A particle A moves to the right with a speed of $0.600c$ relative to the ground. A second particle, B, moves to the right with a speed of $0.700c$ relative to A. Calculate the speed of B relative to the ground.

23 Particle A moves to the left with a speed of $0.600c$ relative to the ground. A second particle, B, moves to the right with a speed of $0.700c$ relative to A. Find the speed of B relative to the ground.

24 A muon travelling at $0.950c$ covers a distance of 2.00 km (as measured by an earthbound observer) before decaying.

a Calculate the muon's lifetime as measured by the earthbound observer.

b Calculate the lifetime as measured by an observer travelling along with the muon.

25 The lifetime of the unstable pion particle is measured to be 2.6×10^{-8} s (when at rest). This particle travels a distance of 20 m in the laboratory just before decaying. Calculate its speed.

In the following questions, the frames S and S′ have their usual meanings, that is, S′ moves past S with velocity v and, when their origins coincide, both clocks are set to zero.

26 In frame S an explosion occurs at position $x = 600$ m and time $t = 2.0\,\mu$s. Frame S′ is moving at speed $0.75c$ in the positive direction. Determine where and when the explosion takes place according to frame S′.

27 Frame S′ moves with speed $0.98c$. An explosion occurs at the origin of frame S′ when the clocks in S′ read 6.0 μs. Calculate where and when the explosion takes place according to frame S.

28 Frame S′ moves with speed $0.60c$. Calculate the reading of the clocks in frame S′ as the origin of S′ passes the point $x = 120$ m.

29 Two events in frame S are such that $\Delta x = x_2 - x_1 = 1200$ m and $\Delta t = t_2 - t_1 = 6.00$ μs.

a **i** The speed v is $0.600c$. Calculate $\Delta x' = x'_2 - x'_1$ and $\Delta t' = t'_2 - t'_1$.

ii Determine whether event 1 could cause event 2.

b Determine whether there is a value of v such that $\Delta t' = 0$. Comment on your answer.

30 Two events in frame S are such that $\Delta x = x_2 - x_1 = 1200$ m and $\Delta t = t_2 - t_1 = 3.0$ μs.

a **i** The speed v is $0.600c$. Calculate $\Delta x' = x'_2 - x'_1$ and $\Delta t' = t'_2 - t'_1$.

ii Determine whether event 1 could cause event 2.

b Determine whether there is a value of v such that $\Delta t' < 0$. Comment on your answer.

31 Two simultaneous events in frame S are separated by a distance $\Delta x = x_1 - x_2 = 1200$ m. Determine the time separating these two events in frame S′, stating which one occurs first. The speed v is $0.600c$.

Learning objectives

- Understand, sketch and work with spacetime diagrams.
- Represent worldlines.
- Explain the twin paradox.

A3 Spacetime diagrams

This section deals with a pictorial representation of relativistic phenomena that makes the concepts of time dilation, length contraction and simultaneity particularly transparent.

A3.1 The invariant hyperbola

The Lorentz transformations have the following important consequence. Consider an event that is measured to have coordinates (x, t) in S and (x', t') in S′. As we know, these coordinates are different in the two frames; the observers disagree about the space and time coordinates. But they all agree on this: the quantity $(x^2 - c^2t^2)$ is the same as $(x'^2 - c^2t'^2)$! We can prove this easily as follows:

$$(x'^2 - c^2t'^2) = \gamma^2(x - vt)^2 - c^2\gamma^2\left(t - \frac{v}{c^2}x\right)^2$$

$$= \gamma^2\left(x^2 - 2xvt + v^2t^2 - \left(c^2t^2 - 2tvx + \frac{v^2}{c^2}x^2\right)\right)$$

$$= \gamma^2\left(x^2\left(1 - \frac{v^2}{c^2}\right) - (c^2 - v^2)t^2\right)$$

$$= \gamma^2\left(x^2\left(1 - \frac{v^2}{c^2}\right) - c^2(1 - \frac{v^2}{c^2})t^2\right)$$

$$= \gamma^2\left(1 - \frac{v^2}{c^2}\right)(x^2 - c^2t^2)$$

$$= x^2 - c^2t^2$$

Exam tip

Remember that $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

and so $\gamma^2\left(1 - \frac{v^2}{c^2}\right) = 1$.

The usefulness and significance of this result will become apparent in the next section.

A3.2 Spacetime diagrams (Minkowski diagrams)

In Topic 2, we saw lots of motion graphs. In particular, we saw graphs of position (vertical axis) versus time (horizontal axis). In relativity it is customary to show these graphs with the axes reversed, that is, with time plotted on the vertical axis and position on the horizontal. These are called **spacetime diagrams**. They are also called **Minkowski diagrams** in honour of H. Minkowski, a mathematician friend of Einstein who helped him with the mathematical formulation of the theory. For reasons we will see shortly, it is very convenient to instead plot ct on the vertical axis rather than time itself. The vertical axis then also has units of length. Since the speed of light is a constant everyone agrees on, knowing the value of ct allows us to find the value of t.

Figure **A.26** shows a spacetime diagram and an event (marked by the dot). The space and time coordinates of the event are read from the graph in the usual way: we draw lines through the dot parallel to the axes and see where the lines intersect the axes. This event has $x = 5.0\,\text{m}$ and $ct = 6.0\,\text{m}$, giving $t = \dfrac{6.0}{3.0\times 10^8} = 2.0\times 10^{-8}\,\text{s}$.

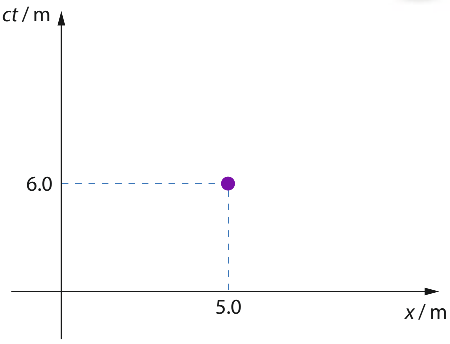


Figure A.26 The space and time coordinates of an event.

Now consider a particle that is at rest at position $x = a$ (Figure **A.27**). As time goes by we may think of a sequence of events showing the position of the particle (which does not change) at different times. This sequence of events traces the straight blue line on the spacetime diagram. This is called the **worldline** of the particle.

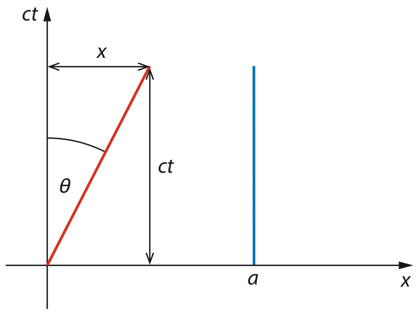


Figure A.27 The worldline of a particle at rest is the sequence of events showing the position of the particle at different times.

Another particle that starts from $x = 0$ at $t = 0$ and moves with constant positive velocity would have the worldline shown in red. The speed of this particle is given by

$$v = \frac{\text{distance}}{\text{time}} = \frac{x}{t} = c\frac{x}{ct} = c\tan\theta$$

so $\tan\theta = \dfrac{v}{c}$.

> The tangent of the angle of the worldline with the ct-axis gives the speed, expressed in units of the speed of light.

This is why it is convenient to plot ct on the vertical axis: a photon moves at speed c and so it makes an angle of $\theta = \tan^{-1}\dfrac{c}{c} = 45°$ with both axes. Since nothing can have a speed that exceeds c, the worldline of any particle will always make an angle less than 45° with the ct-axis. In Figure **A.28**, the red worldline represents a particle moving to the right. The green worldline belongs to a particle moving to the left. The blue worldline is impossible since it involves a speed greater than c.

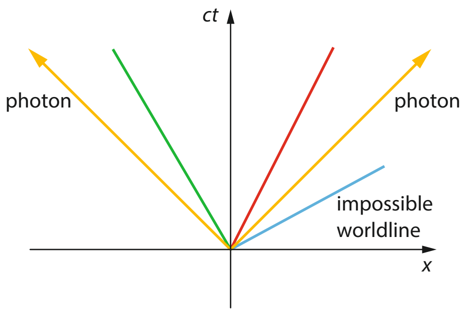


Figure A.28 Various worldlines. The one in blue is impossible because it corresponds to a particle moving faster than light.

Now consider a second inertial reference frame S′ that moves to the right with speed v. Let us assume that when clocks in both frames show zero the origins of the frames coincide. The origin of frame S′ moves to the right with speed v. Therefore, the worldline of the origin of S′ will make an angle $\theta = \tan^{-1}\dfrac{v}{c}$ with the ct-axis of S. But this worldline is just the collection of all events with $x' = 0$, and is therefore the time axis of the frame S′. Because the speed of light is the same in all frames, the space axis of the new frame must make the same angle with the old x-axis. Thus the new frame has axes as shown in red in Figure **A.29**. An event will have different coordinates in the two frames, as we know. To find the coordinates of an event in the frame S′, draw lines parallel to the slanted axes and see where they intersect the axes. The coordinates in the two frames are connected by Lorentz transformations.

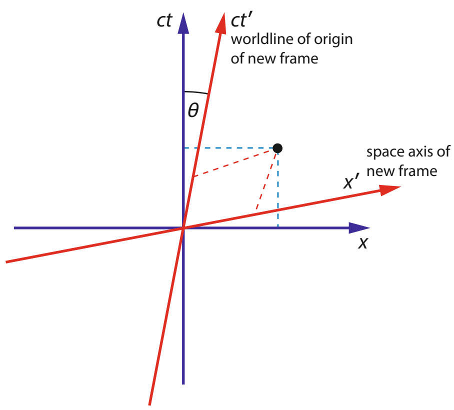


Figure A.29 Two frames with a relative velocity on the same spacetime diagram. The red axes represent a frame moving to the right.

Worked examples

A.17 Use the spacetime diagram below to estimate the speed of frame S′ relative to frame S.

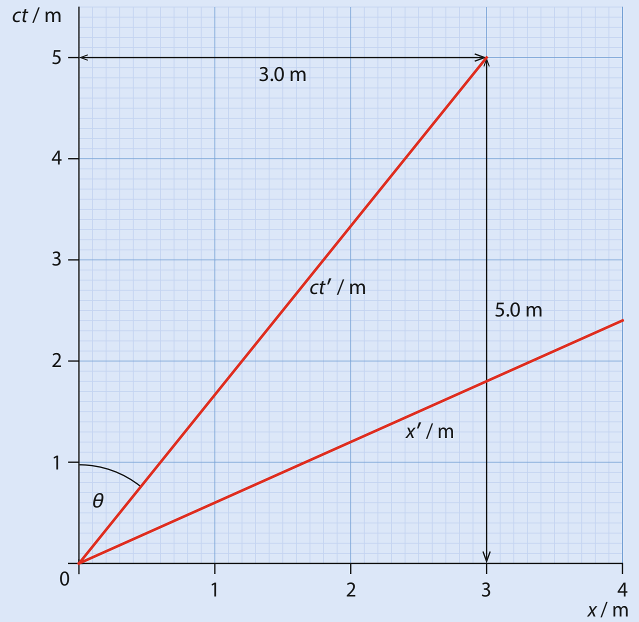


Figure A.30

The tangent of the angle θ is $\tan\theta = \frac{3.0}{5.0} = 0.60$ and so $\frac{v}{c} = 0.60 \Rightarrow v = 0.60c$.

A.18 The spacetime diagram below shows three events. List them from earliest to latest, as observed in each frame.

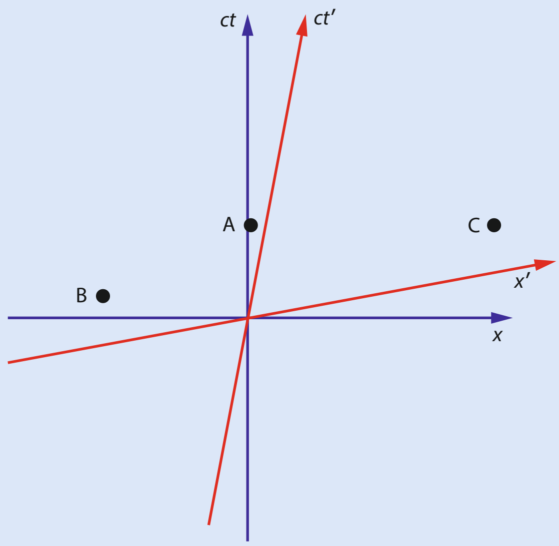


Figure A.31

In S (blue frame), events A and C are simultaneous and occur after event B (A and C are on the same line parallel to the x-axis).

In S′ (red frame), events B and C are simultaneous and occur before event A (B and C are on the same line parallel to the x'-axis).

Let us now see the significance of the equation known as **invariant** hyperbola.

Figure **A.32** shows the invariant hyperbola passing through a point with coordinates ($x=1$, $ct=0$). Its equation is $c^2t^2-x^2=-1$. All events on the blue curve have the same value of $c^2t^2-x^2$, namely -1. Furthermore these events, as observed in frame S′, have the same value of $c^2t'^2-x'^2$, namely -1. If $c^2t^2-x^2$ is plotted on the spacetime axes of frame S, the invariant hyperbola intersects the space axis of frame S′ at the point (i.e. event) P. The coordinates of P are (x', $ct'=0$). Since $c^2t'^2-x'^2=c^2t^2-x^2$, it follows that $c^2t'^2-x'^2=-1$. Since $ct'=0$, it follows that $x'=1$ m.

This result shows that the scales on the two space axes (x and x') are not the same.

This is the basis for length contraction, as we will see in the next section.

The scales are also different on the time axes.

A simpler way to see that the scales on the axes are different is to note in Figure **A.32** that the blue dashed line passes through the event with coordinates $x=1$, $ct=0$ (in S). It is parallel to the ct'-axis, and intersects the x'-axis at Q. The coordinates of Q are clearly $ct'=0$ and

$$x'=\gamma(x-vt)$$
$$=\gamma(1-0)$$
$$=\gamma$$

This sets the scale on the x'-axis.

Consider now the invariant hyperbola that passes through the point with coordinates ($ct=1$, $x=0$). Its equation is $c^2t^2-x^2=1-0=1$). This is plotted in Figure **A.33**.

The hyperbola intersects the time axis of frame S′ at event P, whose coordinates are (ct', 0). Since $(ct')^2-(x')^2=c^2t^2-x^2$, it follows that $(ct')^2-(x')^2=1$, and therefore at event P, $(ct')^2-0=1$ and so $ct'=1$ m. We again see that the scales on the two time axes are different. This is the basis for time dilation.

Again, a simpler way to see the difference in scale is to note in Figure **A.33** that the blue dashed line passes through the event with coordinates $x=0$, $ct=1$ m (in S). It intersects the ct'-axis at Q. Its coordinates are $x'=0$ and

$$t'=\gamma\left(t-\frac{vx}{c^2}\right)$$
$$ct'=\gamma\left(ct-\frac{vx}{c}\right)$$
$$ct'=\gamma(1-0)$$
$$ct'=\gamma$$

This sets the scale on the ct'-axis.

Exam tip
The mathematical details of the invariant hyperbola are not required in examinations. However, you should understand why the scales on the axes of frame S and frame S′ are different. The important thing to remember is that the scales are in fact different.

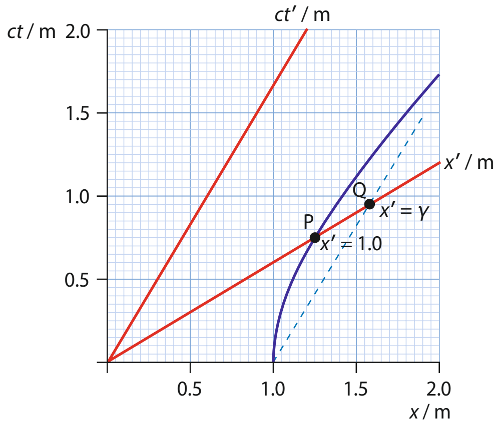


Figure A.32 **a** The invariant hyperbola sets the scale on the x'-axis (red frame). **b** This scale is also set by the dashed blue line, which is parallel to the ct'-axis and passes through $x=1$ m.

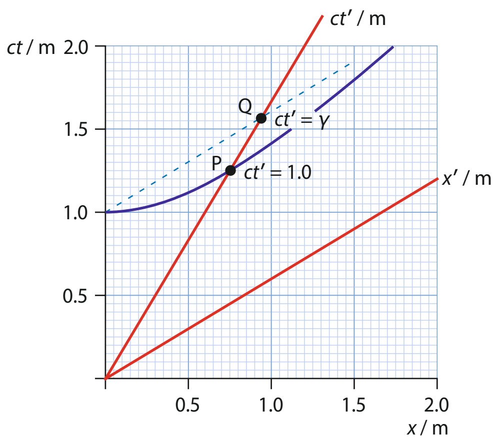


Figure A.33 The invariant hyperbola sets the scale on the ct'-axis (red frame).

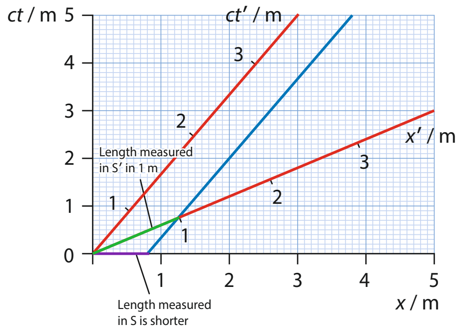


Figure A.34 Rod at rest in S′. Observers in S′ measure a length of 1 m for the rod. Observers in S measure a shorter length.

Exam tip
For observers in S, the rod is moving. So its length must be measured when the ends are recorded at the same time.

Exam tip
Do not be led astray by your knowledge of Euclidean geometry! The length in S′ 'looks' longer, but it isn't. The scales on the two axes are not the same.

A3.3 Length contraction and spacetime diagrams

Imagine a rod of length 1 m at rest in the frame S′. At $t'=0$ the left end of the rod is at $x'=0$ and the other end is at $x'=1$ m. The wordlines of the ends of the rod are shown in Figure **A.34**. The left end of the rod has a worldline that is the ct'-axis; the worldline of the other end is the blue line (which is parallel to the ct'-axis).

The wordlines of the two ends intersect the x-axis at 0 and 0.8 m. These two intersection points represent the positions of the moving rod's ends **at the same time** in frame S. Their difference therefore gives the length of the rod as measured in S. The length is 0.8 m, less than 1 m, the length in S′. The rod which is moving according to observers in S has contracted in length when measured in frame S.

What if we had a rod of proper length 1 m at rest in the frame S which was viewed by observers in the frame S′? This is illustrated in Figure **A.35**.

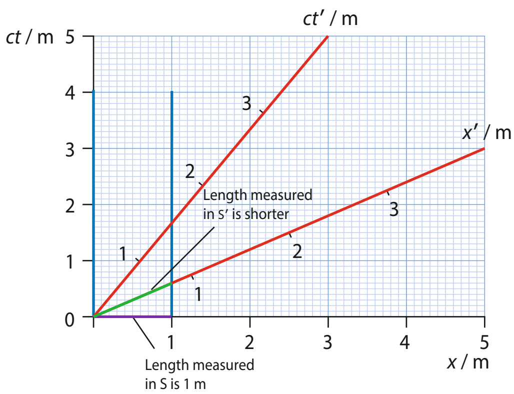


Figure A.35 Rod at rest in S. Observers in S measure a length of 1 m for the rod. The blue lines are the worldlines of the ends of the rod. Observers in S′ measure a shorter length.

The two thick blue lines represent the worldlines of the ends of the rod. They intersect the x'-axis at 0 and 0.8 m. These two intersection points represent the positions of the rod's ends **at the same time** in frame S′. Their difference therefore gives the length of the rod as measured in S′. The length is 0.8 m, less than 1 m, the length in S. The rod which is moving according to observers in S′ has contracted in length when measured in frame S′.

A3.4 Time dilation and spacetime diagrams

Figure **A.36** is a spacetime diagram showing the standard frames S and S′. A clock at rest at the origin of S′ ticks at O and then at P. The time between ticks is $ct'=1$ m. What is the time between ticks according to frame S? We have to draw a line parallel to the x-axis through P. This intersects the S time axis at point Q. This interval is greater than 1. Since P has coordinates ($ct'=1$ m, $x'=0$) in S′, this event in the frame S has time coordinate

$$t=\gamma\left(t'+\frac{vx'}{c^2}\right)=\gamma t'$$

$$ct=\gamma ct'$$

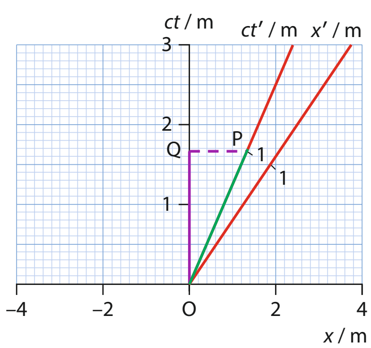


Figure A.36 The green line is the worldline of a clock at rest in frame S′. This clock shows $ct'=1$ at P.

Similarly, Figure **A.37** can be used to analyse a clock at rest at the origin of frame S. It ticks at O and then at P. The time between ticks is such that $ct = 1$. From P we draw a line parallel to the x'-axis, which intersects the ct'-axis at Q. The interval OQ is longer than 1 m.

Exam tip
Do not be led astray by your knowledge of Euclidean geometry! In triangle OPQ the hypotenuse is the side OP and so it would seem to be the longest side. But the geometry of spacetime diagrams is not Euclidean. The Pythagorean theorem does not hold.

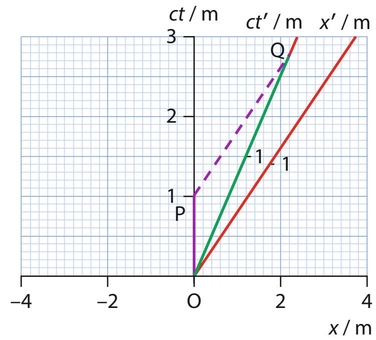


Figure A.37 The purple line is the worldline of a clock at rest in frame S. This clock shows $ct = 1$ at P.

A3.5 The twin paradox

We have seen that the effect of time dilation is symmetric: if you move relative to me, I say that your clocks are running slow compared with mine but you say that my clocks are running slow compared with yours. This symmetry of time dilation is shown clearly in the last two figures in the previous section. An issue arises when two clocks, initially at the same place and showing the same time (say zero), are separated. Suppose one clock moves at a relativistic speed away, suddenly reverses direction and comes back to its starting place next to the clock that stayed behind. The readings of the two clocks are compared. What do they show? In the **twin paradox** version of the story, the clocks are replaced by twins. Jane stays on the Earth and claims that since her twin brother Joe is the one who moved away he must be younger. But Joe can claim that it is Jane and the Earth that moved away and then came back, so Jane should be younger. Who is the younger of the two when the twins are reunited?

Deeply ingrained ideas are hard to get rid of

The physicist W. Rindler, discussing the twin paradox in his book *Introduction to Special Relativity* (Oxford University Press, 1991), says:

> It is quite easily resolved, but seems to possess some hidden emotional content that makes it the subject of interminable debate among the dilettantes of relativity.

So deeply ingrained is the idea of absolute time in us that we find this example hard to accept.

The resolution of the 'paradox' is that Jane has been in the same inertial frame throughout, whereas Joe changed inertial frames in the interval of time it took for him to turn around. He was in an inertial frame moving outwards but he changed to one moving inwards for the return trip. During the changeover from one frame to the other he must have experienced acceleration and forces which Jane never did. The acceleration makes this situation asymmetric, and Joe is the one who is younger on his return to Earth.

To understand this quantitatively, suppose Joe leaves the Earth in a rocket at a speed of $0.6c$, travels to a distant planet 3.0 ly away (as measured by Earth observers) and returns. The gamma factor γ is 1.25. As Jane sees it, his outbound trip takes $\frac{3.0\,\text{ly}}{0.6c} = 5\,\text{yr}$, and another 5 yr to return. She will have aged 10 years when her brother returns. For Joe, time is running slower by the gamma factor, so he will age by $\frac{5.0}{1.25} = 4\,\text{yr}$ on the outward trip and another 4 yr on the way back. He will be 8 years older. We can show all of this on a space–time diagram (Figure **A.38**).

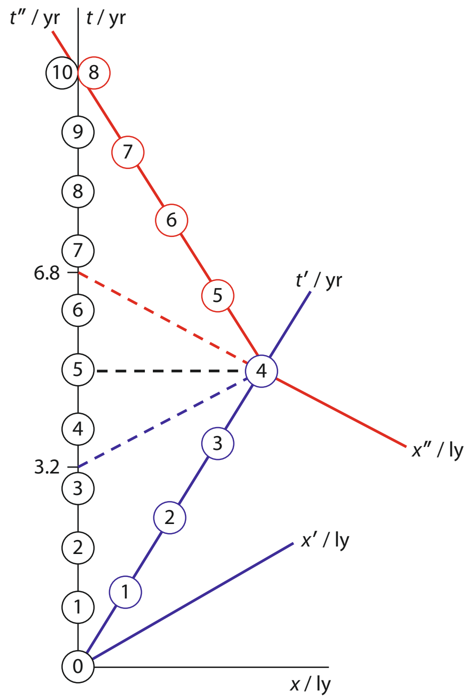


Figure A.38 Spacetime diagram used to resolve the twin paradox. (Adapted from N. David Mermin, *It's About Time*, Princeton University Press, 2005)

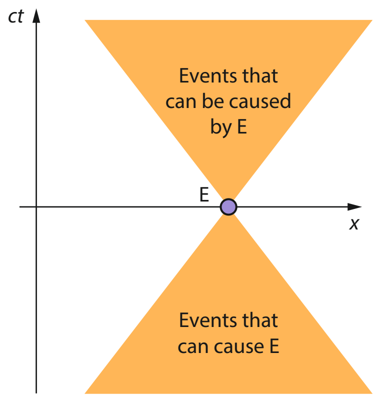


Figure A.39 The past and future light cones of event E.

The black frame, S, is Jane's. The blue frame, S′, is Joe's on the way out, and the red frame, S″, is Joe's on the way back. The circles show what the clocks read in each frame. We see that, at the moment before Joe turns around and changes frame, the Earth clock shows 5 yr and Joe's clock shows 4 yr. When Joe's clock shows 4 yr, he determines (by drawing a line parallel to the x'-axis) that Earth clocks show 3.2 yr. (Time dilation is symmetric!) As soon as Joe changes frame, he determines (by drawing a line parallel to the x''-axis) that Earth clocks show 6.8 yr. The sudden switch from one frame to another seems to create a missing time of $10 - 2 \times 3.2 = 3.6$ yr. But this is not mysterious and nothing strange has happened to either Jane or Joe during this time; it has to do with the fact that we have various frames. The 'now' of frame S′ is found by drawing lines parallel to the x'-axis. The 'now' of frame S″ is found by drawing lines parallel to the x''-axis. Thus, what the two frames understand by 'now' is different.

A3.6 Causality

Figure **A.39** shows an event E on a spacetime diagram. The cones above and below E are defined respectively by the timelines of photons leaving E and arriving at E. The cone formed by the arriving photons (the past light cone) includes all events that could in principle have caused E to occur. Since nothing can move faster than light, no event in the past of E outside the shaded light cone could have influenced E. Similarly, the future light cone of E consists of those events that could in principle be caused by E. E cannot influence any event outside this light cone.

Nature of science

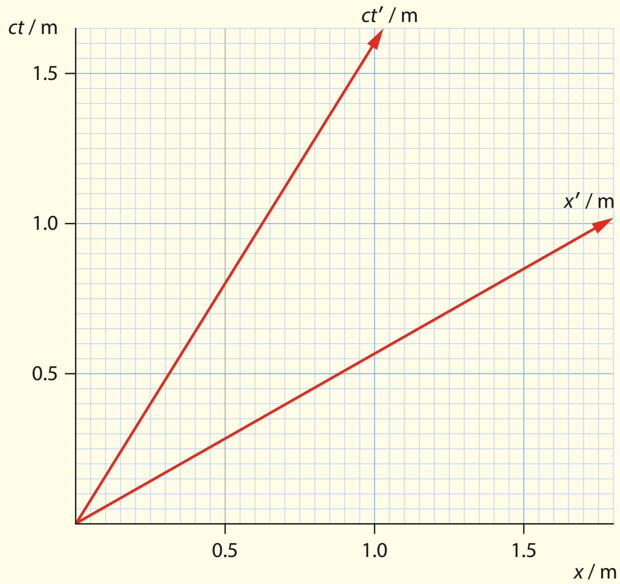
The power of diagrams – again

Spacetime diagrams offer a simple and pictorial way of understanding relativity. They can be used to unambiguously resolve misunderstandings and 'paradoxes'. Their power lies in their simplicity and their clarity. In physics, using diagrams with appropriate notation to describe situations has always helped understanding. Spacetime diagrams are especially useful in showing what event can or cannot cause another. Feynman diagrams are another example of visualisation of a model, as is the use of vectors to show magnitude and direction.

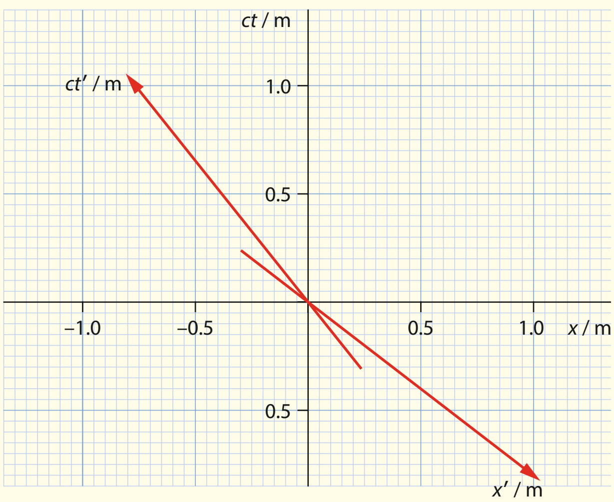
Test yourself

In the questions that follow, the spacetime diagrams represent two inertial frames. The black axes represent frame S. The red axes represent a frame S′ that moves past frame S with velocity v.

32 Use the spacetime diagram to calculate the velocity of frame S′ relative to S.



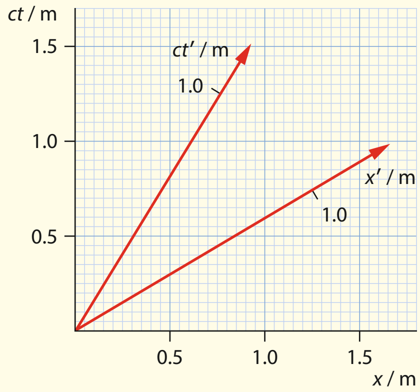
33 Use the space–time diagram to calculate the velocity of frame S′ relative to S.



34 a On a copy of the spacetime diagram, draw the worldline of a particle that is:

i at rest in frame S at $x = 1.0$ m

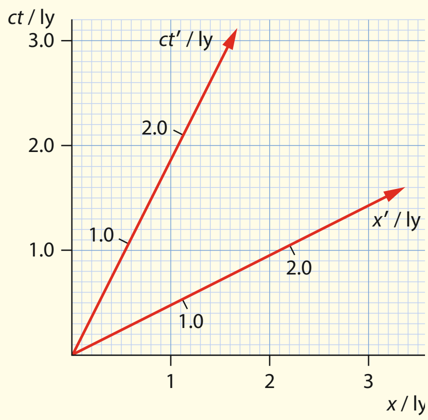
ii moving with velocity $-0.80c$ as measured in S at $x = 1.0$ m at $t = 0$.



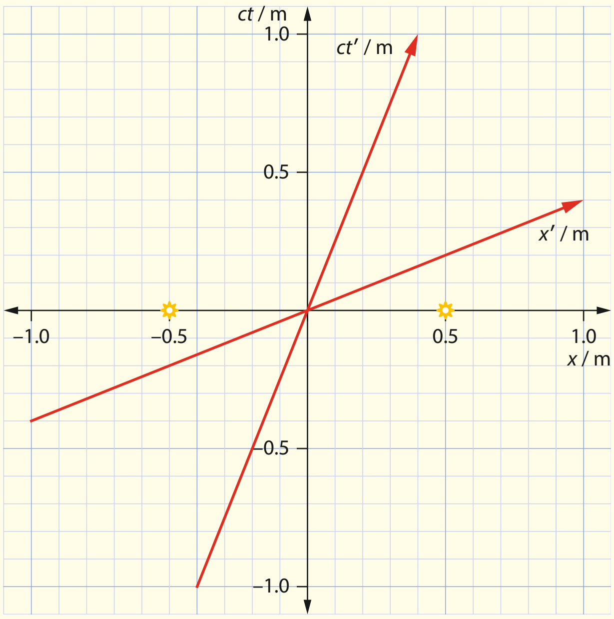
b A photon is emitted from position $x = +1.0$ m at $t = 0$. Use the spacetime diagram from part **a** to estimate when the photon arrives at the eye of an observer at rest at the origin of frame S′, as observed **i** in S and **ii** in S′.

35 a S′ represents an alien attack cruise ship. Use the diagram below to determine the speed of the cruise ship relative to S.

b At $t = 0$, a laser beam moving at the speed of light is launched from $x = +3.0$ ly towards the cruise ship. By drawing the worldline of the laser beam, estimate the time when the beam hits the cruise ship, as observed **i** in S and **ii** in S′.



36 The diagram shows two lamps at $x = \pm 0.5\,\text{m}$ that turn on at $t = 0$ in frame S.

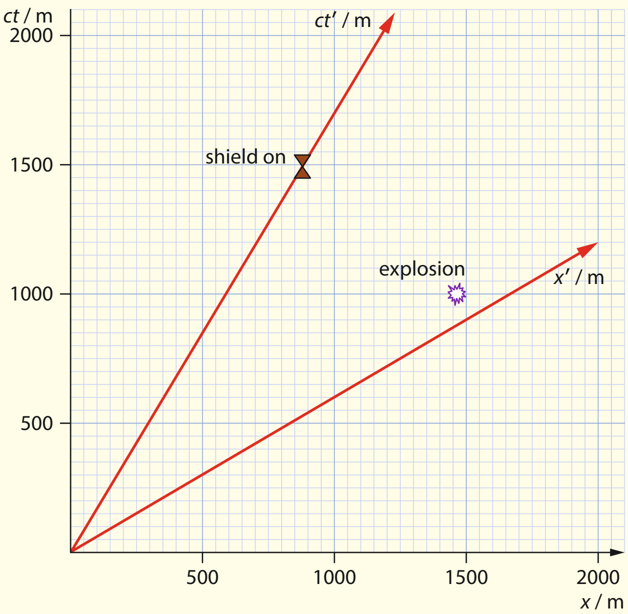


a Determine which lamp turns on first as observed in frame S′.

b Draw the worldlines of the photons emitted from the two lamps towards an observer at rest at the origin of S′.

c Identify the lamp whose light reaches an observer at the origin of frame S′ first.

37 In the spacetime diagram below, the red axes represent a spacecraft that is moving past a planet (black axes). An explosion on the planet takes place at $x = 1500\,\text{m}$ and $ct = 1000\,\text{m}$, emitting deadly photons. A shield around the spacecraft is turned on at the indicated point in spacetime.



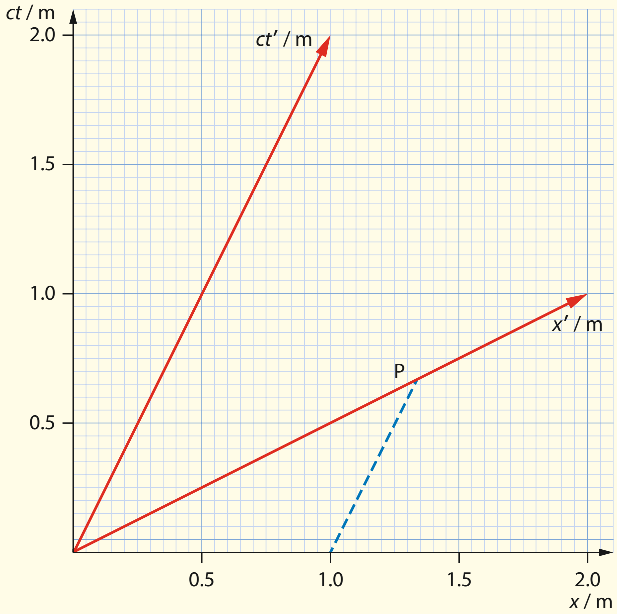
a Determine if the spacecraft will be saved.

b Using Lorentz transformations or otherwise, calculate, in the spacecraft frame:

i the time of the explosion

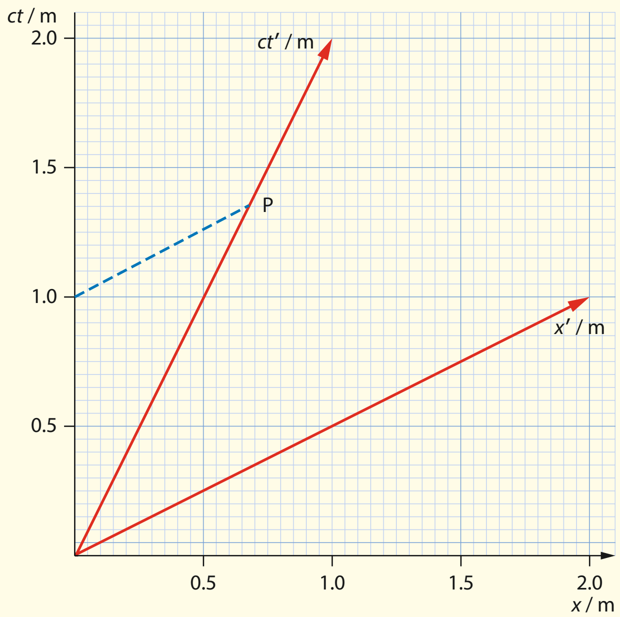
ii the position of the explosion.

38 **a** The dashed blue line in the spacetime diagram below is parallel to the ct'-axis and intersects the x'-axis at P. Using Lorentz transformations, find the coordinates of P in the frame S′, and hence label the event with coordinates ($x' = 1\,\text{m}$, $ct' = 0$).



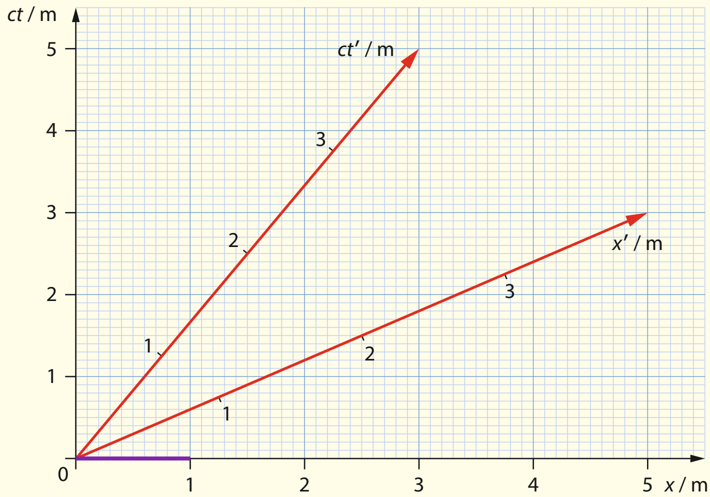
b Repeat part **a**, where now the speed v is arbitrary. Express your answer in terms of the gamma factor, γ.

39 a The dashed blue line in the spacetime diagram is parallel to the x'-axis and intersects the ct'-axis at P. Find the coordinates of P in the frame S′, and hence label the event with coordinates ($x' = 0$, $ct' = 1$ m).



b Repeat part **a**, where now the speed v is arbitrary. Express your answer in terms of the gamma factor γ.

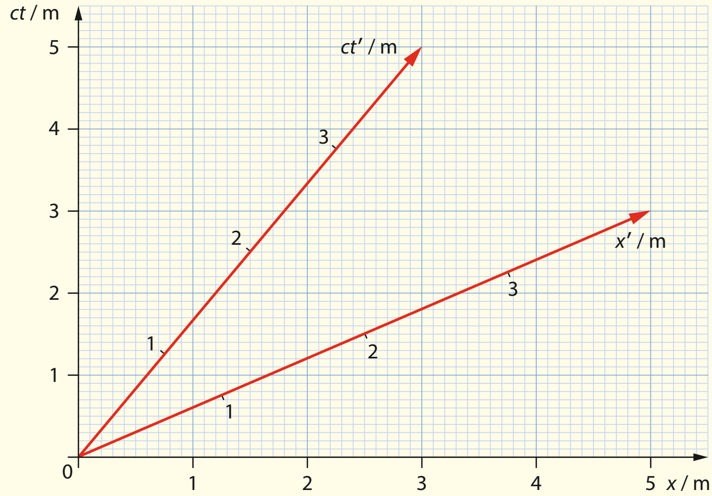
40 The purple line in the spacetime diagram below represents a rod of length 1.0 m at $t = 0$ as measured in the frame S.



a On a copy of this diagram, draw appropriate lines to show that the length of the rod measured in frame S′ is less than 1 m.

b Draw a line to represent a rod of length 1.0 m as measured in the frame S′. By drawing appropriate lines, show that the length of the rod measured in frame S is less than 1 m.

41 In the following spacetime diagram a clock is at rest at the origin of frame S′.



Its first tick occurs at $t' = 0$; mark this event on the diagram. The second tick of the clock occurs when ct' = 1 mas measured in S′; mark this event on the same spacetime diagram. By drawing appropriate lines, estimate the time in between ticks as measured in S.

Learning objectives

- Understand and use the concepts of total and rest energy.
- Work with **relativistic momentum**.
- Solve problems with particle acceleration.
- Appreciate the invariance of electric charge.
- Appreciate photons as massless relativistic particles.
- Work with relativistic units.

Exam tip
The phrase 'to create a particle from the vacuum' refers to particle collisions in particle accelerators in which energy, usually in the form of photons, materialises as particle–antiparticle pairs.

A4 Relativistic mechanics (HL)

This section introduces the changes in Newtonian mechanics that are necessary as a result of the postulates of relativity. We will have to modify the concepts of **total energy** and momentum.

A4.1 Relativistic energy and rest energy

One of the first consequences of Einstein's theories in mechanics is the equivalence of mass and energy. The theory of relativity predicts that, to a particle of mass m that is at rest with respect to some inertial observer, there corresponds an amount of energy E_0 that the observer measures to be $E_0 = mc^2$. (We will not be able to give a proof of this statement in this book.) This energy is called the **rest energy** of the particle.

> Rest energy is the amount of energy needed to produce a particle at rest.

Similarly, if the particle moves with speed v relative to some inertial observer, the energy corresponding to the mass of the particle that this observer will measure is given by

$$E = \gamma mc^2$$

$$= \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This is energy that the particle has because it has mass and because it moves. If the particle has other forms of energy, such as gravitational potential energy or electrical potential energy, then the total energy of the particle will be γmc^2 plus the other forms of energy. We will mostly deal with situations where the particle has no other forms of energy associated with it. The total energy of the particle in that case is then just γmc^2.

It is important to note immediately that, as the speed of the particle approaches the speed of light, the total energy approaches infinity (Figure **A.40**). This is a sign that a particle with mass cannot reach the speed of light. Only particles without mass, such as photons, can move at the speed of light.

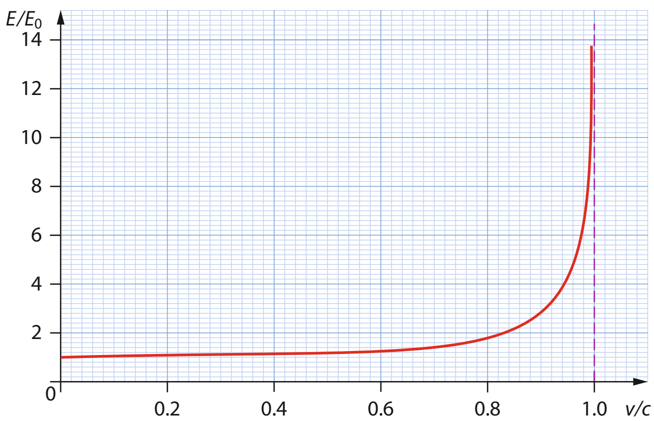


Figure A.40 A graph of the variation with $\frac{v}{c}$ of the ratio of the total energy E to the rest energy E_0 of a particle. As the speed of the particle approaches the speed of light, its energy increases without limit.

Worked examples

A.19 Find the speed of a particle whose total energy is double its rest energy.

We have that

$$E = \gamma mc^2$$

$$2mc^2 = \gamma mc^2$$

$$\Rightarrow \gamma = 2$$

$$\Rightarrow \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{2}$$

$$\Rightarrow 1 - \frac{v^2}{c^2} = \frac{1}{4}$$

$$\Rightarrow \frac{v^2}{c^2} = \frac{3}{2}$$

$$\Rightarrow v = \frac{\sqrt{3}}{2} c = 0.866\,c$$

A.20 Find **a** the rest energy of an electron and **b** its total energy when it moves at a speed equal to $0.800c$.

a The rest energy is

$$E = mc^2$$

$$= 9.1 \times 10^{-31} \times 9 \times 10^{16}\,\text{J}$$

$$= 8.19 \times 10^{-14}\,\text{J}$$

$$= \frac{8.19 \times 10^{-14}}{1.6 \times 10^{-19}}\,\text{eV}$$

$$= 0.511\,\text{MeV}$$

b The gamma factor at a speed of $0.80c$ is

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - 0.8^2}} = \frac{1}{0.6} = 1.667$$

and so the total energy is

$$E = \gamma mc^2 = 1.667 \times 0.511\,\text{MeV} = 0.852\,\text{MeV}$$

A.21 A proton (rest energy 938 MeV) has a total energy of 1170 MeV. Find its speed.

Since the total energy is given by $E = \gamma mc^2$, we have that

$$1170 = \frac{938}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\sqrt{1 - \frac{v^2}{c^2}} = 0.8017$$

$$1 - \frac{v^2}{c^2} = 0.6427$$

$$\frac{v^2}{c^2} = 0.3573$$

$$v = 0.598\,c$$

If a particle is accelerated through a potential difference of V volts, its total energy will increase by an amount qV, where q is the charge of the particle. If the particle is initially at rest, its initial total energy is the rest energy, $E_0 = mc^2$. After going through the potential difference, the total energy will be $E = mc^2 + qV$. We can then find the speed of the particle, as the next example shows.

Worked examples

A.22 An electron of rest energy 0.511 MeV is accelerated through a potential difference of 5.0 MV in a lab.

a Find its total energy with respect to the lab.

b Find its speed with respect to the lab.

a The total energy of the electron will increase by

$$qV = 1e \times 5.0 \times 10^6 \text{ volts} = 5.0\,\text{MeV}$$

and so the total energy is

$$E = m_0c^2 + qV = 0.511\,\text{MeV} + 5.0\,\text{MeV}$$

$$= 5.511\,\text{MeV}$$

b We know that

$$E = \gamma mc^2$$

$$5.511 = \gamma \times 0.511$$

$$\gamma = \frac{5.511}{0.511}$$

$$= 10.785$$

Since $\gamma = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$, it follows that

$$10.785 = \frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$$

$$\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{10.875} (= 0.0927)$$

$$1-\frac{v^2}{c^2} = 0.008597$$

$$v = 0.996\,c$$

A.23 a A proton is accelerated from rest through a potential difference V. Calculate the value of V that will cause the proton to accelerate to a speed of $0.95c$. (The rest energy of a proton is 938 MeV.)

b Determine the accelerating potential required to accelerate a proton from a speed of $0.95c$ to a speed of $0.99c$.

a The gamma factor at a speed of $0.95c$ is

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{1}{\sqrt{1-0.95^2}}$$

$$= 3.20$$

The total energy of the proton after acceleration is thus

$$E = \gamma mc^2$$

$$= 3.20 \times 938\,\text{MeV}$$

$$= 3002\,\text{MeV}$$

From

$$E = mc^2 + qV$$

we find

$$qV = (3002 - 938)\,\text{MeV} = 2064\,\text{MeV}$$

and so

$$V = 2.1 \times 10^9\,\text{V}$$

Notice how we have avoided using SI units in order to make the numerical calculations easier.

b The total energy of the proton at a speed of $0.95c$ is (from part **a**) $E = 3002\,\text{MeV}$. The total energy at a speed of $0.99c$ is (working as in **a**) $E = 6649\,\text{MeV}$. The extra energy needed is then $6649 - 3002 = 3647\,\text{MeV}$, so the accelerating potential must be $3.6 \times 10^9\,\text{V}$. Notice that a larger potential difference is needed to accelerate the proton from $0.95c$ to $0.99c$ than from rest to $0.95c$. This is a sign that it is impossible to reach the speed of light. (See also the next example.)

A.24 A constant force is applied to a particle which is initially at rest. Sketch a graph that shows the variation of the speed of the particle with time for

a Newtonian mechanics

b relativistic mechanics.

In Newtonian mechanics, a constant force produces a constant acceleration, and so the speed increases uniformly without limit, exceeding the speed of light. In relativistic mechanics, the speed increases uniformly as long as the speed is substantially less than the speed of light, and is essentially identical with the Newtonian graph. However, as the speed increases, so does the energy. Because it takes an infinite amount of energy for the particle to reach the speed of light, we conclude that the particle never reaches the speed of light. The speed approaches the speed of light asymptotically. Note that the speed is always less than the corresponding Newtonian value at the same time. Hence we have the graph shown in Figure **A.41**.

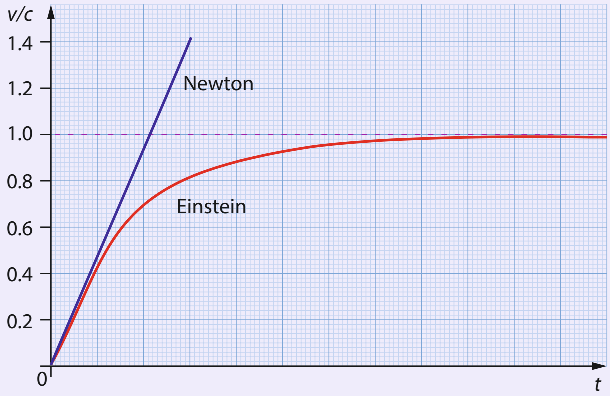


Figure A.41

A4.2 Momentum and energy

In the last section we saw the need to change the definition of total energy. The new definition ensures that a particle cannot be accelerated to the speed of light, because that would take an infinite amount of energy. One other change is required, to momentum, so that in relativistic collisions momentum is conserved as it is in ordinary mechanics.

In classical mechanics, momentum is given by the product of mass and velocity, but in relativity this is modified to

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \gamma m v$$

We still have the usual law of momentum conservation, which states that when no external forces act on a system the total momentum stays the same. The symbol m here stands for the rest mass of the particle and is a constant for all observers.

Unlike Newtonian mechanics, a **constant** force on the particle will produce a **decreasing** acceleration in such a way that the speed never reaches the speed of light.

Worked example

A.25 A constant force F acts on an electron that is initially at rest. Find the speed of the electron as a function of time.

Initially, for small t, the speed increases uniformly, as in Newtonian mechanics. But as t becomes large, the speed tends to the speed of light, but does not reach or exceed it. This is because as the speed increases the acceleration becomes smaller and smaller, and the speed never reaches the speed of light. This results in the graph shown in Figure **A.42**.

Figure A.42

Begin with Newton's second law, $F=\frac{dp}{dt}$, where $p=\gamma mv$ is the momentum of the electron. Since

$$\frac{dp}{dt}=\frac{d}{dt}\left(\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

$$=\frac{m}{\sqrt{1-\frac{v^2}{c^2}}}\frac{dv}{dt}+\left(\frac{v}{c^2}\right)\frac{m}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\frac{dv}{dt}$$

$$\frac{dp}{dt}=\frac{m}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\frac{dv}{dt}$$

it follows that

$$\frac{dv}{dt}=\frac{F}{m}\left(1-\frac{v^2}{c^2}\right)^{3/2}$$

or

$$\frac{dv}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}=\frac{F}{m}dt$$

Integrating both sides, we find that (assuming the mass starts from rest)

$$\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{F}{m}t$$

Solving for the speed, we find

$$v^2=\frac{(Ft)^2}{(mc)^2+(Ft)^2}c^2$$

The way the speed approaches the speed of light is shown in Figure **A.42**.

A4.3 Kinetic energy

A mass moving with velocity v has a total energy E given by

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Its kinetic energy E_k is defined as the total energy minus the rest energy:

$$E_k = E - mc^2$$

Newtonian mechanics is a good approximation to relativity at low speeds

This relativistic definition of kinetic energy does not look similar to the ordinary kinetic energy, $\frac{1}{2}mv^2$. In fact, when v is small compared with c, we can approximate the value of the relativistic factor $\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ using the binomial expansion for $\frac{1}{\sqrt{1 - x}}$ for small x:

$$\frac{1}{\sqrt{1 - x}} \approx 1 + \frac{1}{2}x + \ldots$$

Applying this to E_k with $x = \left(\frac{v}{c}\right)^2$, we find

$$E_k = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\right) - mc^2$$

That is,

$$E_k \approx \tfrac{1}{2}mv^2$$

In other words, for low speeds the relativistic formula reduces to the familiar Newtonian version. For higher speeds, the relativistic formula must be used.

This can be rewritten as

$$E_k = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - mc^2$$

$$= \gamma mc^2 - mc^2$$

$$= (\gamma - 1)mc^2$$

This definition ensures that the kinetic energy is zero when $v = 0$, as can easily be checked. As in ordinary mechanics, the work done by a net force equals the change in kinetic energy in relativity as well.

Total energy, momentum and mass are related: from the definition of momentum, we find that

$$p^2c^2 + m^2c^4 = \frac{m^2v^2c^2}{1-\frac{v^2}{c^2}} + m^2c^4$$

$$= \frac{m^2c^4}{1-\frac{v^2}{c^2}}$$

$$= E^2$$

That is,

$$E = \sqrt{m^2c^4 + p^2c^2}$$

(This formula is the relativistic version of the conventional formula $E = \frac{p^2}{2m}$ from Newtonian mechanics.) This relation can be remembered by using the Pythagorean theorem in the triangle in Figure **A.43**.

The formula we derived above applies also to those particles that have zero mass, such as the photon, in which case $E = pc$. Remembering that, for a photon, $E = hf$, we have $p = \frac{hf}{c} = \frac{h}{\lambda}$.

Exam tip
In the last section we saw, for particle acceleration through a potential difference V, that $qV = \Delta E$. But ΔE is also ΔE_k since, for a particle accelerated from rest, $\Delta E = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2 = \Delta E_k$

A4.4 Relativistic units

We can use units such as $\text{MeV}\,c^{-2}$ or $\text{GeV}\,c^{-2}$ for mass. This follows from the fact that the rest energy of a particle is given by $E_0 = mc^2$ and so allows us to express the mass of the particle in terms of its rest energy as $m = E_0/c^2$. Thus, the statement 'the mass of the pion is $135\,\text{MeV}\,c^{-2}$', means that the rest energy of this particle is $135\,\text{MeV}\,c^{-2} \times c^2 = 135\,\text{MeV}$. (To find the mass in kilograms, we would first have to convert MeV to joules and then divide the result by the square of the speed of light.)

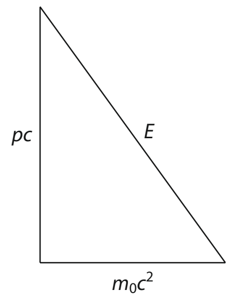


Figure A.43 The rest energy, momentum and total energy are related through the Pythagorean theorem for the triangle shown.

Similarly, the momentum of a particle can be expressed in units of $\text{MeV}\,c^{-1}$ or $\text{GeV}\,c^{-1}$. A particle of rest mass $5.0\,\text{MeV}\,c^{-2}$ and total energy 13 MeV has a momentum given by

$$E^2 = m^2c^4 + p^2c^2$$

$$\Rightarrow p^2c^2 = (169 - 25)\,\text{MeV}^2$$

$$= 144\,\text{MeV}^2$$

$$\Rightarrow pc = 12\,\text{MeV}$$

$$\Rightarrow p = 12\,\text{MeV}\,c^{-1}$$

Exam tip
It is absolutely essential that you are comfortable with these units.

The speed can be found from

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Rightarrow 13 = \frac{5}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Rightarrow \sqrt{1 - \frac{v^2}{c^2}} = \frac{5}{13}$$

$$\Rightarrow \frac{v^2}{c^2} = \frac{144}{169}$$

$$\Rightarrow v = \frac{12}{13}c$$

In conventional SI units, a momentum of $12\,\text{MeV}\,c^{-1}$ is

$$12 \times 10^6 \times 1.6 \times 10^{-19} \frac{\text{J}}{3 \times 10^8\,\text{m}\,\text{s}^{-1}}$$

$$= 6.4 \times 10^{-21} \frac{\text{kg}\,\text{m}^2\,\text{s}^{-2}}{\text{m}\,\text{s}^{-1}}$$

$$= 6.4 \times 10^{-21}\,\text{kg}\,\text{m}\,\text{s}^{-1}$$

Worked examples

A.26 Find the momentum of a pion (rest mass $135\,\text{MeV}\,c^{-2}$) whose speed is $0.80c$.

The total energy is

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \frac{135}{\sqrt{1 - 0.80^2}}$$

$$= 225\,\text{MeV}$$

Using

$$E^2 = m^2c^4 + p^2c^2$$

We obtain

$$pc = \sqrt{225^2 - 135^2}$$

$$= 180\,\text{MeV}$$

$$\Rightarrow p = 180\,\text{MeV}\,c^{-1}$$

A.27 Find the speed of a muon (rest mass = 105 MeV c^{-2}) whose momentum is 228 MeV c^{-1}.

From $p = \gamma m v$, we have $228\,\text{MeV}\,c^{-1} = \gamma \times 105\,\text{MeV}\,c^{-2} \times v$

$$\Rightarrow \gamma \frac{v}{c} = 2.171$$

Hence

$$\frac{1}{1-\left(\frac{v}{c}\right)^2}\left(\frac{v}{c}\right)^2 = 4.715$$

$$\Rightarrow \left(\frac{v}{c}\right)^2 = 4.715 - 4.715\left(\frac{v}{c}\right)^2$$

$$\Rightarrow \left(\frac{v}{c}\right)^2 = \frac{4.715}{5.715}$$

$$= 0.8250$$

and so $v = 0.91c$.

Exam tip
You can also do this by first finding the total energy (251 MeV) and then the gamma factor (2.39) and then the speed.

A.28 Find the kinetic energy of an electron whose momentum is 1.5 MeV c^{-1}.

The total energy of the electron is given by

$$E^2 = m^2c^4 + p^2c^2$$

$$= 0.511\,\text{MeV}^2 + 1.5^2\,\text{MeV}^2\,c^{-2} \times c^2$$

$$= 2.511\,\text{MeV}^2$$

$$\Rightarrow E = 1.58\,\text{MeV}$$

and so

$$E_k = E - mc^2$$

$$= 1.58\,\text{MeV} - 0.511\,\text{MeV}$$

$$= 1.07\,\text{MeV}$$

Nature of science

General principles survive paradigm shifts – usually

Newtonian mechanics relies on basic quantities such as mass, energy and momentum and the laws that relate these quantities. Einstein realised that laws such as energy and momentum conservation are the result of deeper principles and sought to preserve them in relativity. This meant modifying what we normally call energy and momentum in such a way that these quantities are also conserved in relativity.

Test yourself

42 Calculate the energy needed to accelerate an electron to a speed of
a $0.50c$
b $0.90c$
c $0.99c$.

43 **a** Calculate the speed of a particle whose kinetic energy is 10 times its rest energy.
b Calculate the speed of a particle whose total energy is 10 times its rest energy.

44 Find the momentum of a proton whose total energy is 5 times its rest energy.

45 Find the total energy of an electron with a momentum of $350\,\mathrm{MeV}\,c^{-1}$.

46 Determine the momentum, in conventional SI units, of a proton with momentum $685\,\mathrm{MeV}\,c^{-1}$.

47 Find the kinetic energy of a proton whose momentum is $500\,\mathrm{MeV}\,c^{-1}$.

48 The rest energy of a particle is 135 MeV and its total energy is 200 MeV. Find its speed.

49 Calculate the potential difference through which a proton must be accelerated (from rest) so that its momentum is $1200\,\mathrm{GeV}\,c^{-1}$.

50 Find the momentum of a proton whose speed is $0.99c$.

51 Calculate the speed of a proton with momentum $1.5\,\mathrm{GeV}\,c^{-1}$.

52 A proton initially at rest finds itself in a region of uniform electric field of magnitude $5.0 \times 10^6\,\mathrm{V\,m^{-1}}$. The electric field accelerates the proton over a distance of 1.0 km. Calculate:
a the kinetic energy of the proton
b the speed of the proton.

53 **a** Show that the speed of a particle with rest mass m and momentum p is given by

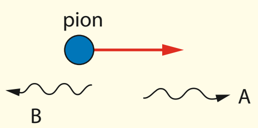
$$v = \frac{pc^2}{\sqrt{m^2c^4 + p^2c^2}}$$

b An electron and a proton have the same momentum. Calculate the ratio of their speeds when the momentum is
i $1.00\,\mathrm{MeV}\,c^{-1}$
ii $1.00\,\mathrm{GeV}\,c^{-1}$.
c What happens to the value of the ratio as the momentum gets larger and larger?

54 A particle at rest breaks apart into two pieces with masses $250\,\mathrm{MeV}\,c^{-2}$ and $125\,\mathrm{MeV}\,c^{-2}$. The lighter fragment moves away at a speed of $0.85c$.
a Find the speed of the other fragment.
b Determine the rest mass of the original particle that broke apart.

55 An electron and a positron, each with kinetic energy 2.0 MeV, collide head-on. The electron and positron annihilate each other, giving two photons.
a Explain why the electron–positron pair cannot create just one photon.
b Explain why the photons must be moving in opposite directions.
c Calculate the energy of each photon.

56 A neutral pion of mass $135\,\mathrm{MeV}\,c^{-2}$ travelling at $0.80c$ decays into two photons travelling in opposite directions, $\pi^0 \to 2\gamma$. Calculate the ratio of the frequency of photon A to that of photon B.



57 Two identical bodies with rest mass 3.0 kg are moving directly towards one another, each with a speed of $0.80c$ relative to the laboratory. They collide and form one body. Determine the rest mass of this body.

58 State the formulas, in terms of the rest mass m of a particle, for
a the relativistic momentum p
b the total energy E.
c Using these formulas, derive the formula $v = \frac{pc^2}{E}$ for the speed v of the particle.
d The formula in **c** applies to all particles, even those that are massless. Deduce that, if the particle is a photon, then $v = c$.

59 **a** Show that when a particle of mass m and charge q is accelerated from rest through a potential difference V, the speed it attains corresponds to a gamma factor $\gamma = 1 + \frac{qV}{mc^2}$.
b A proton is accelerated from rest through a potential difference V. Calculate V such that the proton reaches a speed of $0.998c$.

60 Show that a free electron cannot absorb or emit a photon.

A5 General relativity (HL)

The theory of general relativity was formulated by Albert Einstein in 1915. It is a theory of gravitation that replaces the standard theory of gravitation of Newton, and generalises Einstein's special theory of relativity. The theory of general relativity stands as the crowning achievement of Einstein's genius and is considered to be perhaps the most elegant and beautiful example of a physical theory ever constructed. It is a radical theory in that it relates the distribution of matter and energy in the universe to the structure of space and time. The geometry of spacetime is a direct function of the matter and energy that spacetime contains.

Learning objectives

- Understand the principle of equivalence.
- Understand the bending of light.
- Describe gravitational red-shift and the Pound–Rebka experiment.
- Understand the nature of black holes.
- Understand the meaning of the event horizon.
- Understand time dilation near a black hole.
- Understand the implications of general relativity for the universe as a whole.

A5.1 The principle of equivalence

We saw in Topic **2** that when a person stands on a scale inside a freely falling elevator ('Einstein's elevator') the reading of the scale is zero. It is as if the person is weightless. This is what the scale would read if the elevator were moving at constant velocity in deep space, far from all masses. Similarly, consider an astronaut in a spacecraft in orbit around the Earth. She too feels weightless and floats inside the spacecraft. But neither the person in the falling elevator nor the astronaut is really weightless. Gravity does act on both. We can say that the acceleration of the freely falling elevator or the centripetal acceleration of the spacecraft has 'cancelled out' the force of gravity. The right acceleration can make gravity 'disappear' and make the frame of reference under consideration look like one moving at constant velocity.

The right acceleration can also make gravity 'appear'. Consider an astronaut in a spacecraft that is moving with constant velocity in deep space, far from all masses. The astronaut really is weightless. The spacecraft engines are now ignited and the spacecraft accelerates forward. The astronaut feels pinned down to the floor. If he drops a coin, it will hit the floor, whereas previously it would have floated in the spacecraft. The coin falling to the floor and the sensation of being pinned down are what we normally associate with gravity.

These are two examples where the effects of acceleration mimic those of gravity. Another expression for 'the effects of acceleration' is 'inertial effects'. Einstein elevated these observations to a principle of physics – the **equivalence principle** (EP):

Exam tip
There is hardly an examination paper in which you will not be asked to state this principle.

> Gravitational and inertial effects are indistinguishable.

Applying this principle to the two examples we discussed above, we may re-express it more precisely in two versions:

Version 1: A reference frame moving at constant velocity far from all masses is equivalent to a freely falling reference frame in a uniform gravitational field.

Version 2: An accelerating reference frame far from all masses is equivalent to a reference frame at rest in a gravitational field.

Exam tip
You can use either of these versions of the equivalence principle, but the original statement takes care of both.

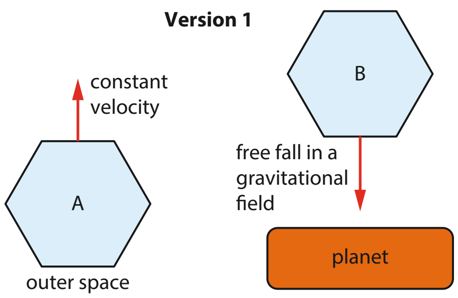


Figure A.44 A frame of reference moving at constant velocity far from any masses (A) and a freely falling frame of reference in a gravitational field (B) are equivalent.

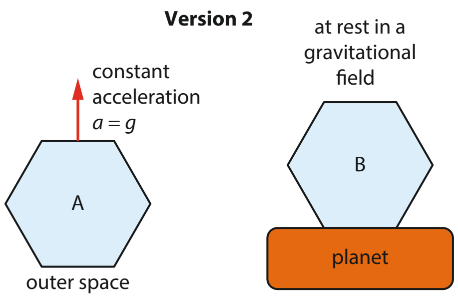


Figure A.45 An accelerating frame of reference far from any masses (A) and a frame at rest in a gravitational field (B) are equivalent.

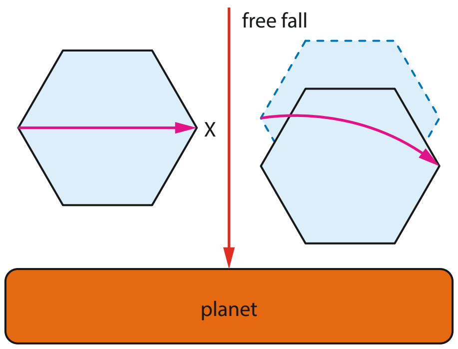


Figure A.46 A ray of light bends towards a massive object.

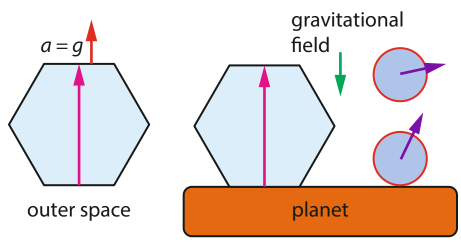


Figure A.47 In the accelerated frame the light ray will have a lower frequency at the top because of the Doppler effect.

Exam tip
In an exam you should be able to use the EP to deduce the bending of light, gravitational red-shift and time dilation.

Figure **A.44** shows a frame of reference A moving at constant velocity in deep space, and a frame of reference B falling freely in a gravitational field. The EP says that the two frames are equivalent. There is no experiment that the occupants of A can perform that will give a different result from the same experiment performed in B, nor can the occupants of either frame decide which of the two states of motion they are 'really' in.

Figure **A.45** shows a frame of reference A accelerating in deep space, and a frame of reference B at rest in a gravitational field. Again, there is no experiment that the occupants of A can perform that will give a different result from the same experiment performed in B, nor can the occupants decide which is which.

This principle has immediate consequences.

A5.2 Consequences of the EP: the bending of light

One consequence of the EP is that **light bends towards a massive body**. Imagine a box falling freely in the gravitational field of a planet (Figure **A.46**). A ray of light is emitted from inside the box initially parallel to the floor. Since, by the EP, this frame is equivalent to a frame moving at constant velocity in outer space, there can be no doubt that the ray of light will hit the opposite side of the box at point X. But an observer, (who must also see the ray hit at X), says that the ray is bending towards the planet.

But this means that, from the point of view of an observer at rest on the surface of the massive body, light has bent towards it.

> A ray of light bends towards a massive body.

Massive objects can thus act as a kind of **gravitational lens**.

A5.3 Consequences of the EP: gravitational red-shift

A second consequence of the EP is that a ray of light emitted upwards in a gravitational field will have its frequency reduced as it climbs higher. In Figure **A.47** a reference frame is at rest in a gravitational field of a planet. A ray of light is emitted upwards. This frame is equivalent to a frame in outer space with an acceleration equal to the gravitational field strength on the planet. In this accelerated frame, an observer at the top of the frame is moving away from the light emitted at the base. So, according to the Doppler effect, he will observe a lower frequency than that emitted at the base. By the EP, the same thing happens in a frame at rest in a gravitational field.

> The frequency of a ray of light is reduced as it moves higher in a gravitational field.

Notice that if the frequency decreases as the ray moves upwards, the period must increase (the speed of light is constant). But the period may be used as a clock. The period is the time in between ticks of a clock. **Gravitational red-shift** is therefore equivalent to saying that we have a gravitational time dilation effect. Consider two identical clocks.

One is placed near a massive body and the other far from it. When the clock near the massive body shows that 1 s has gone by, the faraway clock will show that more than 1 s has gone by.

Time slows down near a massive body.

A5.4 The tests of general relativity

There are many experimental tests that the general theory of relativity has passed. These include:

- The **bending of light** and radio signals near massive objects: this was measured by Eddington in 1919, in agreement with Einstein's prediction.
- Gravitational frequency shift: this was observed by Pound and Rebka in 1960 (see Section **A5.5**).
- The Shapiro time delay experiment: radio signals sent to a planet or spacecraft and reflected back to the Earth take slightly longer to return when they pass near the Sun than when the Sun is out of the way. This delay is predicted by the theory of relativity.
- The precession of the perihelion of Mercury: Mercury has an abnormality in its orbit that Newtonian gravitation cannot account for. Einstein's theory correctly predicts this anomaly.
- Gravitational lensing: massive galaxies may bend light from distant stars and even galaxies, creating multiple images. This has been observed; see Figure **A.48**.

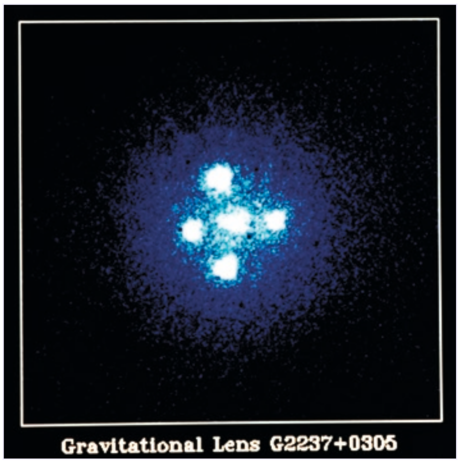


Figure A.48 Multiple images of a distant quasar, caused by an intervening galaxy whose gravitational field has deflected the light from the quasar.

A5.5 The Pound–Rebka experiment

The phenomenon of gravitational red-shift was experimentally verified by the **Pound–Rebka experiment** in 1960. In this experiment, performed at Harvard University, a beam of gamma rays of energy 14.4 keV from a nuclear transition in iron-57 was emitted from the top of a tower 22.6 m high and detected at ground level; see Figure **A.49**. Just as the frequency of light decreases as the light moves upwards in a gravitational field, it increases if the light is moving downwards: we have a blue-shift (which is what Pound and Rebka measured).

Theory predicts that the frequency shift Δf between the emitted and received frequencies is given by

$$\frac{\Delta f}{f} = \frac{gH}{c^2}$$

Here, f stands for the emitted or the observed frequency and H is the height from which the photons are emitted.

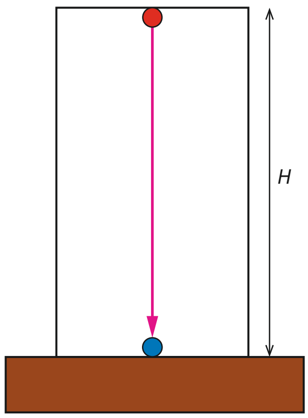


Figure A.49 A photon is emitted from the top of a tower and observed at the base, where its frequency is measured and found to be higher than that emitted.

Worked example

A.29 A photon of energy 14.4 keV is emitted from the top of a 30 m-tall tower towards the ground. Calculate the shift in frequency expected at the base of the tower.

On emission, the photon has a frequency given by

$$E = hf$$

$$\Rightarrow f = \frac{E}{h}$$

$$= \frac{14.4 \times 10^3 \times 1.6 \times 10^{-19}}{6.63 \times 10^{-34}}$$

$$= 3.475 \times 10^{18}\,\text{Hz}$$

The shift (it is a blue-shift) is thus

$$\frac{\Delta f}{f} = \frac{gH}{c^2}$$

$$\Rightarrow \Delta f = f\frac{gH}{c^2}$$

$$= \frac{10 \times 30}{9 \times 10^{16}} \times 3.475 \times 10^{18}\,\text{Hz}$$

$$= 1.16 \times 10^4\,\text{Hz}$$

Note how sensitive this experiment actually is: the shift in frequency is only about 10^4 Hz, compared with the emitted frequency of about 10^{18} Hz, a fractional shift of only

$$\frac{10^4\,\text{Hz}}{10^{18}\,\text{Hz}} = 10^{-14}$$

A5.6 The structure of the theory

The theory of general relativity is a physical theory different from all others, in that:

> The mass and energy content of space determines the geometry of that space and time. The geometry of spacetime determines the motion of mass and energy in the spacetime.

That is,

geometry ⇔ mass–energy

Spacetime is a four-dimensional world with three space coordinates and one time coordinate. In the absence of any forces, a body moves in this four-dimensional world along paths of shortest length, called **geodesics**.

If a single mass M is the only mass present in the universe, the solutions of the Einstein equations imply that, far from this mass, the geometry of space is the usual Euclidean flat geometry, with all its familiar rules (for example, the angles of a triangle add up to 180°). As we approach the neighbourhood of M, the space becomes curved, as illustrated in

Credit where credit is due

Einstein's general theory of relativity would have been impossible to develop were it not for 19th-century mathematicians who were bold enough to question Euclid's fifth axiom of geometry. Modifying that axiom meant that new kinds of geometry became available. Developing the mathematical machinery to describe these geometries was one of the greatest achievements of 19th-century mathematics.

Figure **A.50**. The rules of geometry then have to change. Large masses with small radii produce extreme bending of the spacetime around them. Thus, the motion of a planet around the Sun is, according to Einstein, not the result of a gravitational force acting on the planet (as Newton would have it) but rather due to the curved geometry in the space and time around the Sun created by the large mass of the Sun.

Similarly, light retains its familiar property of travelling from one place to another in the shortest possible time. Since the speed of light is constant, this means that light travels along paths of shortest length: geodesics. In the flat Euclidean geometry we are used to, geodesics are straight lines. In the curved non-Euclidean geometry of general relativity, something else replaces the concept of a straight line. A ray of light travelling near the Sun looks bent to us because we are used to flat space. But the geometry near the Sun is curved and the 'bent' ray is actually a geodesic: it is the 'straight' line appropriate to that geometry.

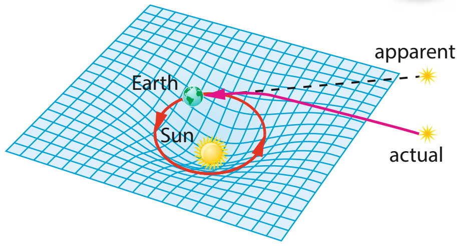


Figure A.50 The mass of the Sun causes a curvature of the space around it. Light from a star bends on its way to the Earth and this makes the star appear to be in a different position from its actual position.

A5.7 Black holes

The theory of general relativity also predicts the existence of objects that contract under the influence of their own gravitation, becoming ever smaller. No mechanism is known for stopping this collapse, and the object is expected to become a hole in spacetime, a point of infinite density. This creates an immense bending of spacetime around this point. This point is called a **black hole**, a name coined by John Archibald Wheeler (Figure **A.51**), since nothing can escape from it. Massive stars can, under appropriate conditions, collapse under their own gravitation and end up as black holes (see Option D, Astrophysics). Powerful theorems by Stephen Hawking and Roger Penrose show that the formation of black holes is inevitable and not dependent too much on the details of how the collapse itself proceeds.

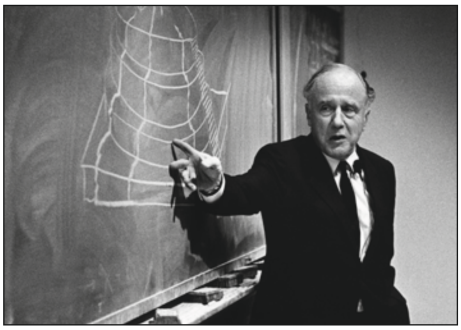


Figure A.51 John A. Wheeler, a legendary figure in black-hole physics and the man who coined the term.

A5.8 The Schwarzschild radius and the event horizon

A distance known as the **Schwarzschild radius** of a black hole is of importance in understanding the behaviour of black holes. Karl Schwarzschild (Figure **A.52**) was a German astronomer who provided the first solution of the Einstein equations.

Figure A.52 Karl Schwarzschild was the man who provided the first solution of Einstein's equations.

The Schwarzschild radius is given by

$$R_S = \frac{2GM}{c^2}$$

where M is the mass and c the speed of light. This radius is not the actual radius of the black hole (the black hole is a point), but the distance from the hole's centre that separates space into a region from which an object can escape and a region from which no object can escape (Figure **A.53**).

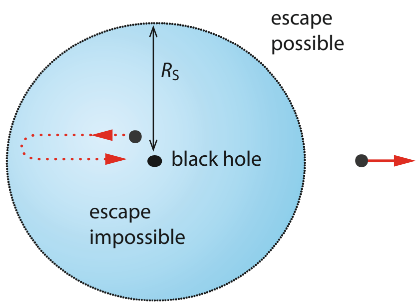


Figure A.53 The Schwarzschild radius splits space into two regions. From within this radius nothing – not even light – can escape.

Quantum black holes are different from classical black holes

The statement that nothing escapes from a black hole is true only if we ignore quantum effects. Stephen Hawking showed in 1974 that, when quantum effects are taken into account, black holes radiate just like a black body does, with the reciprocal of the mass playing the role of temperature. For massive black holes this is not significant, since in that case the temperature is very small. However, it is interesting that, once the concept of 'temperature' was introduced, knowledge of thermodynamics could be transferred to black-hole physics in a formal way, even though black holes do not have a temperature in the normal sense.

Any object closer to the centre of the black hole than R_S will fall into the hole; no amount of energy supplied to this body will allow it to escape from the black hole.

The escape velocity at a distance R_S from the centre of the black hole is the speed of light, so nothing can escape from within this radius. The Schwarzschild radius is also called the **event horizon radius**. The latter name is apt, since anything taking place within the event horizon cannot be seen by or communicated to the outside. The area of the event horizon is taken as the surface area of the black hole (but, remember, the black hole is a point).

The Schwarzschild radius can be derived in an elementary way without recourse to general relativity if we assume that a photon has a mass m on which the black hole's gravity acts. Then, as in the calculation of escape velocity in Topic **10** on gravitation,

$$\frac{1}{2}mc^2 = \frac{GM}{R_S}m$$

Thus m cancels and we can solve for R_S.

It can be readily calculated from this formula that, for a star of one solar mass ($M \approx 2 \times 10^{30}$ kg), the Schwarzschild radius is about 3 km:

$$R_S = \frac{2GM}{c^2}$$

$$= \frac{2 \times 6.67 \times 10^{-11} \times 2 \times 10^{30}}{(3 \times 10^8)^2}$$

$$\approx 3 \times 10^3\,\text{m}$$

This means, for example, that if the Sun were to become a black hole its entire mass would be confined within a sphere with a radius of 3 km or less. For a black hole of one Earth mass, this radius would be just 9 mm.

Figure **A.54** shows that an observer on the surface of a star that is about to become a black hole would only be able to receive light through a cone that gets smaller and smaller as the star approaches its Schwarzschild radius. This is because rays of light coming 'sideways' will bend towards the star and will not reach the observer.

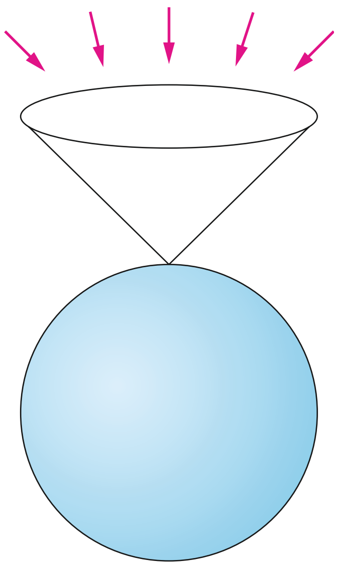


Figure A.54 The event horizon on a collapsing star is rising; that is, only rays within a steadily shrinking vertical cone can reach an observer on the star.

A5.9 Time dilation in general relativity

We have seen that the EP predicts that time runs slow near massive bodies. A special case of this phenomenon takes place near a black hole.

Consider a clock that is at a distance r from the centre of a black hole of Schwarzschild radius R_S (the clock is outside the event horizon, i.e. $r > R_S$); see Figure **A.55**. An observer who is stationary with respect to the clock measures the time interval between two ticks of the clock to be Δt_0. An identical clock very far away from the black hole will measure a time interval

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{R_S}{r}}}$$

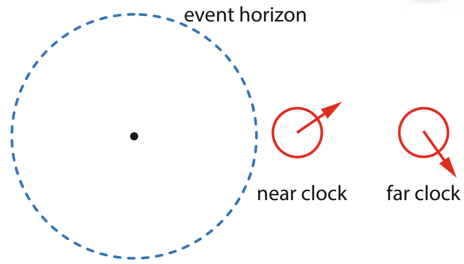


Figure A.55 A clock near a black hole runs slowly compared with a clock far away.

This formula is the general relativistic analogue of time dilation in special relativity. It applies near a black hole.

Thus, consider a (theoretical) observer approaching a black hole. This observer sends signals to a faraway observer in a spacecraft, informing the spacecraft of his position. When his distance from the centre of the black hole is $r = 1.50R_S$, the observer stops and sends two signals 1 s apart (as measured by his clocks). The spacecraft observers will receive signals separated in time by

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{R_S}{r}}}$$

$$= \frac{1.00}{\sqrt{1 - \frac{1}{1.50}}}$$

$$= 1.73\,\text{s}$$

The extreme case of time dilation is when the observer is just an infinitesimally small distance outside the event horizon. If he stops there and sends two signals 1 s apart, the faraway observer will receive the signals separated by an enormous interval of time. In particular, if the observer is *at* the event horizon when the second signal is emitted, that signal will never be received.

A5.10 General relativity and the universe

Soon after he published his general theory of relativity, Einstein applied his equations for general relativity to the universe as a whole. The result looks like this:

$$R_{\mu\nu} - (\tfrac{1}{2}R - \Lambda)g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

and will not be examined!

On a large scale, the universe looks like a cloud of dust of density ρ. Under various simplifying assumptions, the equations reduce to an equation for a quantity that we will loosely call the 'radius' of the universe, R. Solving these equations gives the dependence of R on time. Einstein himself believed in a static universe, which would have R = constant. His calculations, however, did not give a constant R.

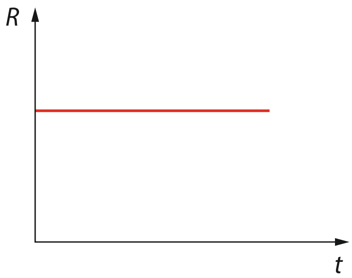


Figure A.56 A model of the universe with a constant radius. Einstein introduced the cosmological constant in order to make the universe static.

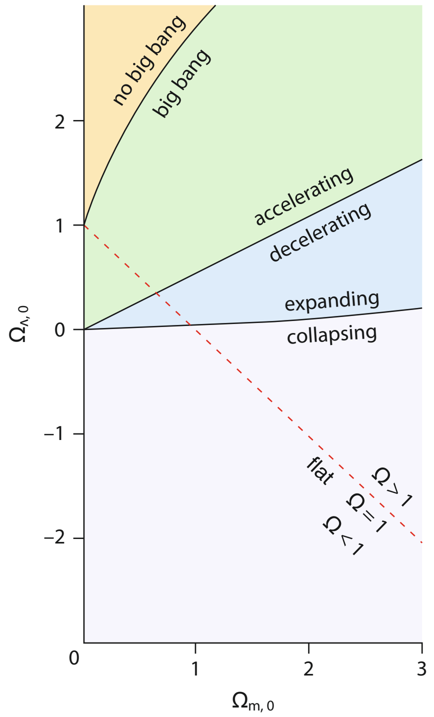


Figure A.57 There are various possibilities for the evolution of the universe, depending on how much energy and mass it contains. (From M. Jones and R. Lambourne, *An Introduction to Galaxies and Cosmology*, Cambridge University Press, in association with The Open University, 2004. © The Open University, used with permission)

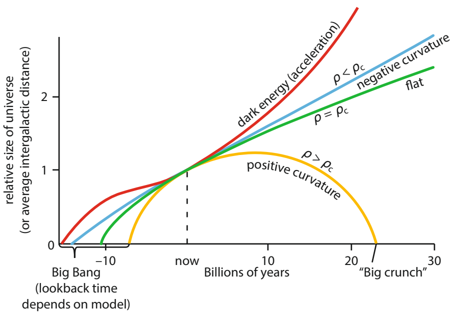


Figure A.59 Solutions of Einstein's equations for the evolution of the scale factor. The present time is indicated by 'now'. Notice that the age of the universe varies depending on which solution is chosen – in other words, different solutions imply different ages. Three models assume zero dark energy; the red line does not.

So he modified them, adding the famous **cosmological constant** term Λ. This term made R constant! This is shown in Figure **A.56**.

In this model there is no 'Big Bang' and the universe always has the same size. This was before Hubble discovered that the universe is expanding. Einstein missed the chance to theoretically predict an expanding universe; he later called the addition of the cosmological constant 'the greatest blunder of my life'. This constant may be thought of as being related to a vacuum energy, energy that is present in all space. This energy is now called **dark energy**. The cosmological constant went into obscurity for many decades but it did not go away: it was to make a comeback with a vengeance much later!

The first serious attempt to find how R depends on time was made by the Russian mathematician Alexander Friedmann (1888–1925), in work later taken up by Lemaître, Robertson and Walker. Friedmann applied the Einstein equations and realised that there were a number of possibilities: the solutions depend on how much matter and energy the universe contains (Figure **A.73**).

We define the **density** parameter for matter, Ω_m, and for dark energy, Ω_Λ, as

$$\Omega_\text{m} = \frac{\rho_\text{m}}{\rho_\text{crit}} \text{ and } \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_\text{crit}}$$

where ρ_m is the actual density of matter in the universe and ρ_crit is a reference density called the critical density, about $10^{-26}\,\text{kg}\,\text{m}^{-3}$; ρ_Λ is the density of dark energy. The Friedmann equations give various solutions, depending on the values of Ω_m and Ω_Λ. Deciding which solution to pick depends crucially on these values, which is why cosmologists have expended enormous amounts of energy and time trying to accurately measure Ω_m and Ω_Λ. In Figure **A.57** the subscript 0 indicates the values of these parameters at the present time. There are four main regions in this diagram. Models above the red dashed line are ones in which the geometry of the universe resembles the surface of a sphere (Figure **A.58c**). Points below the line have a geometry like that of the surface of a saddle (Figure **A.58a**). Points on the line imply a flat universe where the rules of Euclidean geometry apply (Figure **A.58b**).

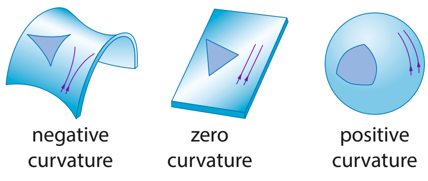


Figure A.58 Three models of the universe with different curvature. **a** In negative-curvature models, the angles of a triangle add up to less than 180° and initially parallel lines eventually diverge. **b** Ordinary flat geometry. **c** Positive curvature, in which the angles of a triangle add up to more than 180° and initially parallel lines eventually intersect. (From M. Jones and R. Lambourne, *An Introduction to Galaxies and Cosmology*, Cambridge University Press, in association with The Open University, 2004. © The Open University, used with permission)

Figure **A.59** shows how the radius R of the universe varies with time for various values of the parameters.

In all cases the radius starts from zero, implying a 'Big Bang'. In one possibility, $R(t)$ starts from zero, increases to a maximum value and then decreases to zero again – the universe collapses after an initial period of expansion. This is called the **closed** model, and corresponds to $\Omega_\text{m} > 1$, that is, $\rho_m > \rho_{crit}$. A second possibility applies when $\Omega_m < 1$, that is, $\rho_m < \rho_{crit}$. The present data from the Planck satellite observatory indicate that $\Omega_m \approx 0.32$ and $\Omega_\Lambda \approx 0.68$, so $\Omega_m + \Omega_\Lambda \approx 1$, on the red dashed line in Figure **A.57**. This implies a flat universe, meaning that at present our universe has a flat geometry, 32% of its mass–energy content is matter, 68% is dark energy and it is expanding forever at an accelerating rate. This is shown by the red line in Figure **A.59**.

Nature of science

Creative and critical thinking

The theory of general relativity is an almost magical theory in which the energy and mass content of spacetime determines the geometry of spacetime. In turn, the kind of geometry spacetime has determines how the mass and energy of the spacetime move about. To connect these ideas in the theory of general relativity, Einstein used intuition, creative thinking and imagination. Initial solutions of Einstein's equations suggested that no light could escape from a black hole. Then a further imaginative leap – the development of quantum theory – showed an unexpected connection between black holes and thermodynamics.

Test yourself

61 Discuss the statement 'a ray of light does not actually bend near a massive object but follows a straight-line path in the geometry of the space around the massive object'.

62 A spacecraft filled with air at ordinary density and pressure is far from any large masses. A helium-filled balloon floats inside the spacecraft, which now accelerates to the right. Determine which way (if any) the balloon moves.

63 The spacecraft of question **62** now has a lighted candle in it. The craft accelerates to the right. Discuss what happens to the flame of the candle.

64 **a** Describe what is meant by the equivalence principle.

b Explain how this principle leads to the predictions that:

i light bends in a gravitational field

ii time 'runs slower' near a massive object.

65 Describe what the general theory of relativity predicts about a massive object whose radius is getting smaller.

66 In a reference frame falling freely in a gravitational field, an observer attaches a mass *m* to the end of a spring, extends the spring and lets the mass go. He measures the period of oscillation of the mass. Discuss whether he will find the same answer as an observer doing the same thing:

a on the surface of a very massive star

b in a true inertial frame far from any masses.

67 Calculate the shift in frequency of light of wavelength 500.0 nm emitted from sea level and detected at a height of 50.0 m.

68 A collapsed star has a radius that is 5.0 times larger than its Schwarzschild radius. An observer on the surface of the star carries a clock and a laser. Every second, the observer sends a short pulse of laser light of duration 1.00 ms and wavelength 4.00×10^{-7} m (as measured by her instruments) towards another observer in a spacecraft far from the star. Discuss qualitatively what the observer in the spacecraft can expect to measure for:

a the wavelength of the pulses

b the frequency of reception of consecutive pulses

c the duration of the pulses.

69 Two identical clocks are placed on a rotating disc. One is placed on the circumference of the disc and the other halfway towards the centre. Explain why the clock on the circumference will run slowly relative to the other clock.

70 Calculate the density of the Earth if its entire mass were confined within a radius equal to its Schwarzschild radius.

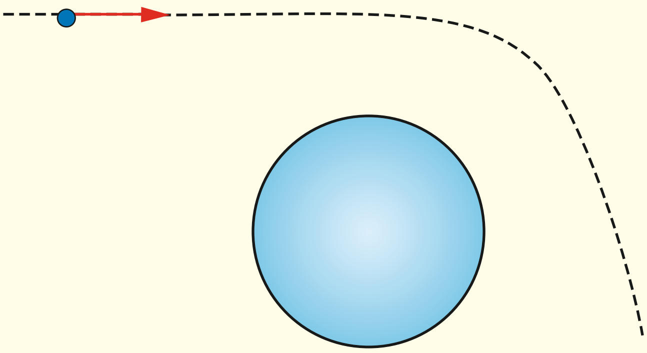
71 Calculate the Schwarzschild radius of a black hole with a mass equal to 10 solar masses.

72 Explain what is meant by **geodesic**.

73 A mass m moves past a massive body along the path shown below. Explain the shape of the path according to:

a Newtonian gravity

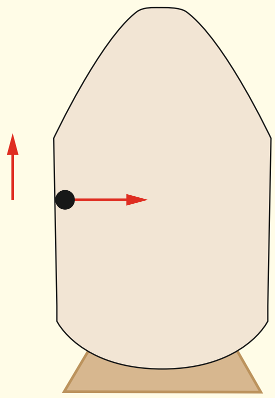
b Einstein's theory of general relativity.



74 A ball is thrown with a velocity that is initially parallel to the floor of a spacecraft, as shown below. Draw and explain the shape of the ball's path when:

a the spacecraft is moving with constant velocity in deep space, far from any large masses

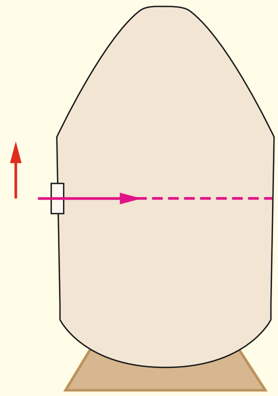
b the spacecraft is moving with constant acceleration in deep space, far from any large masses.



75 A ray of light parallel to the floor of a spacecraft enters the spacecraft through a small window, as shown below. Draw and explain the shape of the ray's path when:

a the spacecraft is moving with constant velocity in deep space, far from any large masses

b the spacecraft is moving with constant acceleration in deep space, far from any large masses.



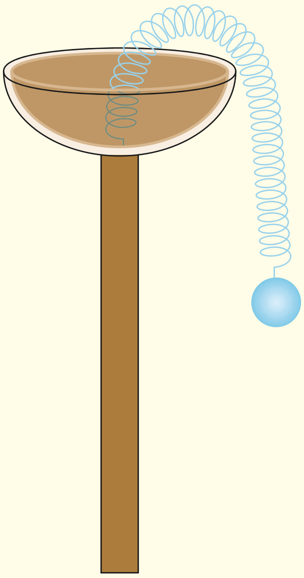
76 An observer is standing on the surface of a massive object that is collapsing and is about to form a black hole. Describe what the observer sees in the sky:

a before the object shrinks past its event horizon

b after the object goes past its event horizon.

77 A plane flying from southern Europe to New York City will fly over Ireland, across the North Atlantic, over the east coast of Canada and then south to New York. Suggest why such a path is followed.

78 **Einstein's birthday present**. A colleague of Einstein at Princeton presented him with the following birthday present. A long tube was connected to a bowl at its top end. A spring was attached to the base of the bowl and connected to a heavy brass ball that hung out of the bowl. The spring was very 'weak' and could not pull the ball into the bowl. The exercise was to find a sure-fire method for putting the ball into the bowl without touching it. Einstein immediately found a way of doing it. Can you?



79 An observer approaching a black hole stops and sends signals to a faraway spacecraft every 1.0 s as measured by his clocks. The signals are received 2.0 s apart by observers in the spacecraft. Determine how close he is to the event horizon.

80 A spacecraft accelerates in the vacuum of outer space far from any masses. An observer in the spacecraft sends a radio message to a stationary spacecraft far away. The duration of the radio transmission is 5.0 s, according to the observer's clock in the moving spacecraft. Explain whether the transmission will take less than, exactly or more than 5.0 s when it is received by the faraway stationary spacecraft.

81 A black hole has a mass of 5.00×10^{35} kg.

a State what is meant by a **black hole**.

b Calculate the Schwarzschild radius R_S of the black hole.

c Explain why the Schwarzschild radius is not in fact the actual radius of the black hole.

d Blue light of frequency 7.50×10^{14} Hz is emitted by a source that is stationary at a distance of $0.10\,R_S$ above the event horizon of a black hole. Calculate the period of this blue light according to an observer next to the source.

e Determine the frequency measured by a distant observer who receives light emitted by this source.

82 **a** State the formula for the gravitational (Schwarzschild) radius R_S of a black hole of mass M.

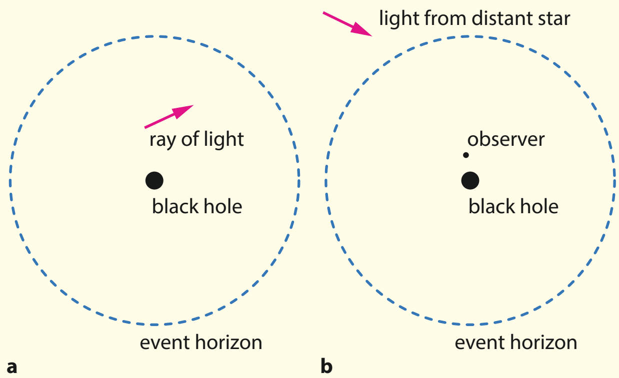
b The sphere of radius R_S around a black hole is called the **event horizon**. State the area of the event horizon of a black hole of mass M.

c Suggest why, over time, the area of the event horizon of a black hole always increases.

d In physics, there is one other physical quantity that always increases with time. Can you state what that quantity is? (You will learn a lot of interesting things if you pursue the analogy implied by your answers to **c** and **d**.)

83 **a** A ray of light is emitted from within the event horizon (dashed circle) of a black hole, as shown below. Copy the diagram and draw a possible path for this ray of light.

b Light from a distant star arrives at a theoretical observer within the event horizon of a black hole, as shown in part **b** of the figure. Explain how it is possible for the ray shown to enter the observer's eye.

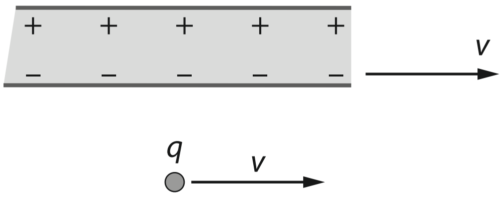


Exam-style questions

1 **a** State what is meant by a **reference frame**. **[2]**

b Discuss the speed of light in the context of Maxwell's theory. **[2]**

c A positive charge q moves parallel to a wire that carries electric current. The velocity of the charged particle is the same as the drift velocity of the electrons in the wire.



Discuss the force experienced by the charged particle from the point of view of an observer:

i at rest with respect to the wire. **[2]**

ii at rest with respect to the charged particle. **[2]**

2 A rocket moves past a space station at a speed of $0.98c$. The proper length of the space station is 480 m. The origins of the space station and rocket frames coincide when the rocket is moving past the middle of the space station. At that instant, clocks in both frames are set to zero.

a State what is meant by **proper length**. **[1]**

b Determine the length of the space station according to an observer in the rocket. **[2]**

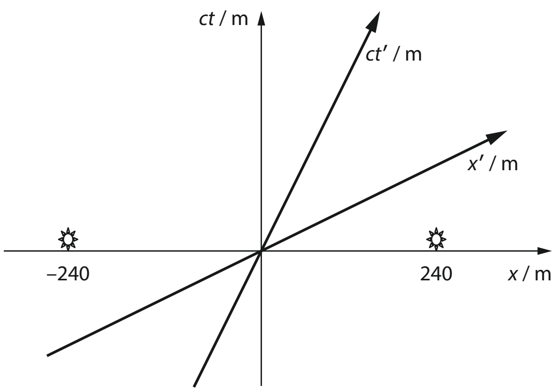
c Two lamps at each end of the space station turn on simultaneously according to space station observers, at time zero.

Determine

i which lamp turns on first, according to an observer in the rocket **[4]**

ii the time interval between the lamps turning on, according to an observer in the rocket. **[3]**

d The spacetime diagram below applies to the frames of the space station and the rocket. The thin axes represent the space station reference frame.



Use the diagram to verify your answer to **c i**. **[1]**

e By drawing appropriate lines on a copy of the spacetime diagram, show the events:

i 'light from the lamp at $x = 240$ m reaches the rocket' (the rocket is assumed to be a point) **[2]**

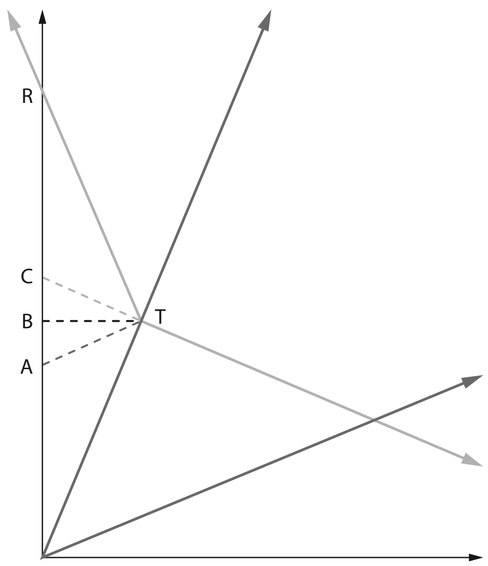
ii 'light from the lamp at $x = -240$ m reaches the rocket' (the rocket is assumed to be a point). **[2]**

3 A rocket moves to the right with speed 0.78*c* relative to a space station. The rocket emits two missiles, each at a speed of 0.50*c* relative to the rocket. One missile, R, is emitted to the right and the other, L, to the left.

a Calculate the velocity of missile
  i R relative to the space station **[2]**
  ii L relative to the space station **[2]**
  iii R relative to missile L. **[2]**

b Show by explicit calculation that the speed of light is independent of the speed of its source. **[2]**

4 The spacetime diagram below may be used to discuss the twin paradox. The thin black axes represent the reference frame of the twin on the Earth. The dark grey axes are for the frame of the travelling twin on her way away from the Earth. The light grey axes represent her homeward frame. The time axes in each frame may be thought of as the worldlines of clocks at rest in each frame. The travelling twin moves at 0.80*c* on both legs of the trip. She turns around at event T when she reaches a planet 12 ly away, as measured by the Earth observers. She returns to Earth at event R.



a Describe what is meant by the **twin paradox**. **[2]**

b Outline how the paradox is resolved. **[2]**

c Calculate the following clock readings:
  i the Earth clock at B **[1]**
  ii the outgoing clock at T **[2]**
  iii the Earth clock at A **[2]**
  iv the Earth clock at C **[2]**
  v the Earth clock at R **[1]**
  vi the incoming clock at R. **[1]**

d State by how much each twin has aged when they are reunited. **[2]**

5 A rocket of proper length 690 m moves with speed $0.85c$ past a lab. A particle is emitted from the back of the rocket and is received at the front. The particle has speed $0.75c$ as measured in the rocket frame.

a Calculate the time it takes for the particle to reach the front of the rocket according to
i observers in the rocket **[1]**
ii observers in the lab. **[3]**

b Suggest whether either of the times calculated in **a** is a proper time interval. **[2]**

c Calculate the speed of the particle according to the lab. **[2]**

d **i** Without using Lorentz transformations, determine the distance travelled by the particle according to the lab. **[2]**
ii Verify your answer to **i** by using an appropriate Lorentz transformation. **[2]**

6 A spacecraft leaves the Earth at a speed of $0.60c$ towards a space station 6.0 ly away (as measured by observers on the Earth).

a Calculate the time the spacecraft will take to reach the space station, according to
i Earth observers **[1]**
ii spacecraft observers. **[2]**

b As the spacecraft goes past the space station it sends a radio signal back towards the Earth.

Calculate the time it will take the signal to arrive on the Earth according to
i Earth observers **[2]**
ii spacecraft observers. **[4]**

7 The average lifetime of a muon as measured in the muon's rest frame is about 2.2×10^{-6} s. Muons decay into electrons. A muon is created at a height of 3.0 km above the surface of the Earth (as measured by observers on the Earth) and moves towards the surface at a speed of $0.98c$. The gamma factor γ for this speed is 5.0. By appropriate calculations, explain how muon decay experiments provide evidence for
i time dilation **[3]**
ii length contraction. **[3]**

HL **8** **a** A proton is accelerated from rest through a potential difference of 2.5 GV.

Determine, for the accelerated proton,
i the total energy **[2]**
ii the momentum **[2]**
iii the speed. **[3]**

b In a hypothetical experiment, two particles, each of rest mass $135\,\mathrm{MeV}c^{-2}$ and moving in opposite directions with speed $0.98c$ relative to the laboratory, collide and form a single particle. Calculate the rest mass of this particle. **[3]**

HL **9** **a** A neutral pion with rest energy 135 MeV c^{-2} and moving at $0.98c$ relative to a lab decays into two photons. One photon is emitted in the direction of motion of the pion and the other in the opposite direction.

i State what is meant by the **rest energy** of a particle. **[1]**

ii Calculate the momentum and total energy of the pion as measured in the lab. **[3]**

b By applying the laws of conservation of energy and momentum to this decay, determine:

i the energy of the forward-moving photon **[3]**

ii the momentum of the backward-moving photon. **[2]**

c **i** Show that the speed of a particle with momentum p and total energy E is given by $v = \frac{pc^2}{E}$. **[2]**

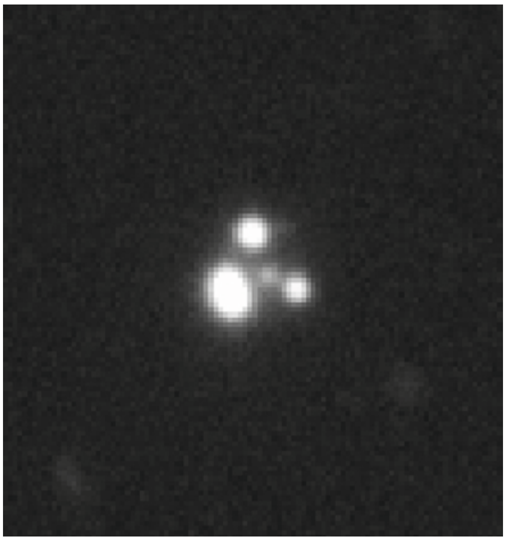
ii Deduce that a particle of zero rest mass must move at the speed of light. **[2]**

HL **10** **a** Describe what is meant by the equivalence principle. **[2]**

b **i** Use the equivalence principle to deduce that a ray of light bends towards a massive body. **[3]**

ii Einstein would claim that the ray does not in fact bend. By reference to the curvature of space, suggest what Einstein might mean by this. **[2]**

c The image below shows a multiple image of a distant quasar.



The image has been formed because light from the quasar went past a massive galaxy on its way to the Earth.

Explain how the multiple image is formed. **[2]**

HL **11** **a** State what is meant by gravitational red-shift. **[2]**

b Explain gravitational red-shift using the equivalence principle. **[4]**

c In the Pound–Rebka experiment, a gamma ray was emitted from the top of a tower 23 m high.

i Calculate the fractional change in frequency observed at the bottom of the tower. **[2]**

ii Explain why, for this experiment to be successful, frequency must be measured very precisely. **[2]**

iii Outline why this experiment is evidence for gravitational time dilation. **[2]**

d The experiment in **c** is repeated in an elevator of height 23 m that is falling freely above the Earth. The gamma ray is emitted from the top of the elevator. Predict the frequency of the gamma ray as measured at the base of the elevator. **[3]**

HL **12** **a** State what is meant by:

i a **black hole** **[1]**

ii the **event horizon** of a black hole. **[1]**

b Calculate the event horizon radius for a black hole of mass 5.0×10^{35} kg. **[2]**

c Suggest why the event horizon radius of a black hole is likely to increase with time. **[2]**

d A probe near the event horizon of the black hole in **b** sends signals to a spacecraft far from the hole every 5.0 s, according to clocks on the probe.

i The signals are received every 15 s, according to clocks on a spacecraft. Determine the distance of the probe from the black hole. **[2]**

ii State **one other** difference that observers on the spacecraft will notice about the received signal. **[1]**